MCMC METHODS FOR PERIODIC AR-ARCH MODELS

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Abstract

Many economic time series reveal periodic and seasonal patterns which can be modeled by periodic AR processes. Using a conjugate normal-gamma model we suggest to model seasonal data by a hierarchical prior distribution. We extend this approach to include periodic ARCH models which we call a PAR-GARCH model. A Metropolis-Hastings step is used for the ARCH part of the model and in case when the exact likelihood function is given.

Keywords: Bayesian periodic AR-ARCH models, hierarchical models.

1 INTRODUCTION

Periodic time series have become increasingly important in many applied sciences and this has given new impact to develop different estimation methods. In this paper we will concentrate on the Bayesian estimation of time series models which is motivated by seasonal econometric applications. In a recent text book Franses (1998) has described the classical estimation of these models while an early Bayesian approach by hierarchical seasonal models can be found in Polasek (1984).

We will briefly describe the hierarchical model for seasonal time series models which follows the hierarchical Gaussian model and the embedding of period models into multivariate time series models. Also we show that this approach can be extended to periodic ARCH (autoregressive conditional heteroskedasticity) or GARCH (generalized ARCH, see Bollerslev (1986)) models and the estimation method we suggest is the MCMC approach which is now widely used in practice for hierarchical models. While we will concentrate on periodic AR (PAR) models it is readily seen that the model range can be extended to include ARMA processes, i.e. a moving average (MA) component.

Section 2 introduces the periodic AR model by specifying a hierarchical prior structure. Section 3 describes the estimation procedure and section 4 introduces periodic AR-ARCH processes.The last section concludes. The appendix contains all the technical details as how to run a Gibbs sampler for a hierarchical regression model and how to use MCMC methods to sample from the posterior distribution which is based on the exact likelihood function for AR models.

2 BAYESIAN PERIODIC AR MODELS

In this section we describe the Bayesian periodic AR model with period (or season) S and explain how we can can specify a prior distribution for the AR coefficients of a PAR model. We consider a time series with seasonal periodicity of length S and for each season we specify a Gaussian regression model. Several ways to introduce a hierarchical prior structure have been suggested over the last decades and for regression models many applications can be found in the literature (see e.g. Polasek (1984) for seasonal time series applications and Gelfand et al. (1990) for the first Gibbs sampling application for hierarchical models).

We will investigate two strategies for PAR models: the first model is the hierarchical pooling model where we make the assumption that the PAR coefficients come from a common hyper-population.

$$\beta_s \sim N[\beta, \Sigma], \quad s = 1, ..., S. \tag{1}$$

The second model introduces smoothness restrictions between neighboring seasons, i.e.

$$\beta_s - \beta_{s-1} \sim N[0, \Sigma], \quad s = 1, ..., S \tag{2}$$

where we make a cyclical restriction on the seasonal regression coefficients $\beta_0 = \beta_S$.

To understand the 'periodic' approach consider the quarterly $PAR(1)$ model:

$$\begin{aligned} y_{1t} &= \beta_{10} + \beta_{11} y_{4,t-1} + u_{1t}, & (3)\\ y_{2t} &= \beta_{20} + \beta_{21} y_{1t} + u_{2t}, & (4)\\ y_{3t} &= \beta_{30} + \beta_{31} y_{2t} + u_{3t}, & (5)\\ y_{4t} &= \beta_{40} + \beta_{41} y_{3t} + u_{4t} & (6) \end{aligned}$$

where we assume that the error term is homoskedastic iid or normally distributed: $u_{4t} \sim N[0, \sigma^2]$. Generalizing, the PAR(1) model with period S is defined as

$$y_{st} = x'_{st}\beta_s + u_{st}, \quad s = 1, ..., S \tag{7}$$

where the regressors for the quarterly $PAR(1)$ model are defined by the vectors $x_{st} = (1, y_{s-1,t})'$, $s = 2, ..., S$, and $x_{1t} = (1, y_{4,t-1})'$.

3 THE HIERARCHICAL PAR MODEL

As we see from the quarterly $PAR(1)$ model, the periodic AR(p) model has more parameters than a non-periodic model and therefore it seems quite reasonable to make some similarity assumptions for the coefficients. A straightforward approach to do this is by specifying a Bayesian hierarchical structure. Assuming a Gaussian (normal) distribution for the data, the hierarchical PAR model is given by

$$y_{st} \sim N[x'_{st}\beta_s, \sigma_s^2 I_n],$$

$$s = 1, ..., S, t = 1, ..., T. \tag{9}$$

and the prior distributions are specified as

$$\beta_s \sim N[\beta, \Sigma], \beta \sim N[b_*, H_*],$$

$$\Sigma^{-1} \sim W[(\gamma_* S_*)^{-1}, \nu_*],$$

$$\sigma_s^{-2} \sim Ga_2[\sigma_*^2, n_*],$$

where W stands for the Wishart distribution and Ga_2 for a gamma-2 distribution, i.e. stands for the re-parameterisation $Ga_2[a, b] = Ga[a/2, b/2]$. We adopt the convention that all hyper-parameters indexed with a $'*'$ are known and have to be specified before the data are known. The estimation of this model is possible by Gibbs sampling and is described in Gelfand et al.(1990).

3.1 The smoothness PAR model

For the Gaussian smoothness PAR(p) model of order p we assume a diagonal regression system like the quarterly PAR(p) model which has the following form

$$\begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \\ y_{4t} \end{pmatrix} = X_t \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix} + \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \\ u_{4t} \end{pmatrix}$$

$$\text{with} \quad X_t = \begin{pmatrix} x'_{1t} & 0 & 0 & 0 \\ 0 & x'_{2t} & 0 & 0 \\ 0 & 0 & x'_{3t} & 0 \\ 0 & 0 & 0 & x'_{4t} \end{pmatrix} \tag{13}$$

and the residual is normally distributed: $u_t \sim N[0, \Sigma]$. For general number of seasons S the PAR model is

$$y_t = X_t \beta + u_t \tag{14}$$

with $X'_t = (x_{1t}, ..., x_{St})$, and $x_{st} = (1, x_{s,t-1}, ..., x_{s,t-p})'$. The dimension of the vector and matrices are $y_t : (S \times 1), X_t : (S \times Sk)$, and

$$\beta : (Sk \times 1), u_t : (S \times 1). \tag{15}$$

Assuming Gaussian distributions for the location parameters and a Wishart distribution for the covariance matrix, the hierarchical model is given by

$$\begin{aligned} y_t &\sim N[X_t \beta, \Omega], \quad , \\ A_k \beta &\sim N[0, \sigma_*^2 I_S \otimes I_k], \\ \Omega &\sim W[(\nu_* \Omega_*)^{-1}, \nu_*] \end{aligned}$$

where the differencing matrix A_k is defined as

$$A = \begin{pmatrix} I_k & 0 & 0 & -I_k \\ -I_k & I_k & 0 & 0 \\ 0 & -I_k & I_k & 0 \\ 0 & 0 & -I_k & I_k \\ \end{pmatrix} \tag{16}$$

and I_k is the identity matrix of order k. Note that the variance parameter σ_*^2 controls the tightness of the smoothness restriction in the prior distribution. a small σ_*^2 will enforce the equality of the β_s coefficients.

4 PERIODIC AR-ARCH MODELS

A further extension of PAR models covers volatile financial time series if we specify an ARCH structure or an GARCH (generalized ARCH) structure for the conditional variances. Using Gaussian distributions we assume a two-stage hierarchical model which is given by

$$\begin{aligned} y_{st} &\sim N[x'_{st}\beta_s, z'_{st}\gamma_s], \\ \beta_s &\sim N[b_*, H_*], \\ \gamma_s &\sim N[\gamma_*, G_*]. \end{aligned}$$

A 3-stage model in analogy to (2) would also be possible. In this model we assume a prior distribution which is the same for all seasons (we will call this also an equi-periodic prior distribution). In Polasek and Jin (1997) we have suggested a tightness prior structure for the AR coefficients which is the case if we assume $b_* = 0$ and $H_*^{-1} = diag(0, 1, 2, ..., k)$ for the prior of the AR coefficients and in similar way we assume for the ARCH coefficients $\gamma_* = 0.01$ and $G_* = H_*$. (The prior mean .01 can be varied with the dimension of the PAR system and the number of ARCH parameters.) The tightness structure is expressed through the variances which shrink in a linear way around the prior location, which is 0 for the AR coefficients and .01 for the ARCH coefficients.

The model specifies for each season s an univariate ARCH model where the prior distribution restricts the coefficients to the stationarity region and shrinks them to zero or a small positive value.

The regression vector is $x'_{st} = (x_{st0}, ..., x_{stk})$ a vector of length $k+1$ and could consist of past time series data or other regression variables. In case of an AR(k) model the regression vector is given by $x'_{st} = (1, y_{s,t-1}, ..., y_{s,t-k})$ and z_{st} is a vector of length $p + q + 1$ for the ARCH(p,q) coefficients, i.e.

$$z'_{st} = (1, h_{s,t-1}, ..., h_{s,t-p}, \varepsilon^2_{s,t-1}, ..., \varepsilon^2_{s,t-q}) \tag{17}$$

with $h_{st} = z'_{st}\gamma_s$ and $\varepsilon_{st} = y_{st} - x'_{st}\beta_s$.

For the MCMC algorithm we have to work out the full conditional distributions (f.c.d.'s). The f.c.d. for the AR coefficients are given by

$$p(\beta_s|\theta\backslash\beta_s, Y) \propto N[b_{s**}, \quad H_{s**}] \tag{18}$$

with the hyper-parameters

$$\begin{aligned} H^{-1}_{s**} &= H^{-1}_* + X'_s D^{-1}_s X_s, \\ b_{s**} &= H_{s**}(H^{-1}_* b_* + X'_s D^{-1}_s y_s), \end{aligned} \tag{19}$$

where the weight matrix is $D_s = diag(z'_{s1}\gamma_s, ..., z'_{sT}\gamma_s)$ with $X'_s = (x_{s1}, ..., x_{sT})$ and $y'_s = (y_{s1}, ..., y_{sT})$.

The f.c.d. for the ARCH parameters has to be generated by a Metropolis step. The full conditional posterior distribution is given by

$$\begin{aligned} p(\gamma_s \mid \theta\backslash\gamma_s, Y) \propto \prod_{t=1}^{T} (z'_{st}\gamma_s)^{-1/2} \\ exp[-\frac{1}{2}(y_{st} - x'_{st}\beta_s)^2 / z'_{st}\gamma_s] \\ \cdot \exp[-\frac{1}{2}(\gamma_s - \gamma_*)' G^{-1}_* (\gamma_s - \gamma_*)]. \end{aligned}$$

We suggest to use as a proposal distribution the least squares regression from the squared residuals of the AR process on the variables in z_t:

$$\hat{\varepsilon^2_{st}} = z'_{st}\gamma_s + u_{st} \quad \text{with} \quad \hat{\varepsilon}_{st} = y_{st} - x'_{st}\beta_s \tag{20}$$

from where we get the parameters for the proposal distribution

$$p(\gamma_s) = N[\hat{\gamma}_s, \hat{G}_s] \tag{21}$$

with $\hat{\gamma}_s = (Z'_s Z_s)^{-1} Z'_s \hat{\varepsilon}_s$ and $\hat{G}_s = \hat{\tau}^2_s (Z'_s Z_s)^{-1}$ where $\hat{\tau}^2_s = \frac{1}{T}\sum_{t=1}^{T} \hat{u}^2_{st}$ and $Z'_s = (z_{s1}, ..., z_{sT})$.
Assuming a smoothness prior we can define a smoothness PAR-PARCH model. This would extend the 2-stage model in (4) to a 3-stage PAR(k)-ARCH(p,q) model:

$$\begin{aligned} y_{st} &\sim N[x'_{st}\beta_s, z'_{st}\gamma_s], \\ \beta_s \sim N[b_*, H_*], \gamma_s &\sim N[\gamma_*, G_*], \\ A_k\beta &\sim N[0, \sigma^2_* I_S \otimes I_k], \\ A_{p'}\gamma &\sim N[0, \tau^2_* I_S \otimes I_p] \end{aligned}$$

where $\gamma' = (\gamma'_1, ..., \gamma'_S)$ and the differencing matrices A_k and $A_{p'}$ (of appropriate dimension $p' = p + q + 1$) are defined as in (17).

5 SUMMARY

The paper has given a brief introduction into MCMC methods for periodic AR models. We have described a hierarchical model for seasonal PAR models and we have shown how PAR models can be extended to PAR-GARCH models and the MCMC algorithm can be used as well. The MCMC approach is very flexible and can be extended to multivariate models or to models with outliers (see Polasek and Jin (1997)). A further open problem is a computational feasable approach to Bayesian model selection.

6 Appendix: The Metropolis Algorithm

Consider a conditional distribution $p(\phi \mid y)$ which is specified up to a constant and cannot be sampled directly (also called target distribution). Then the Metropolis algorithm will create a sequence of samples which converge to the target distribution.

1. Draw a point θ^0 from a starting distribution $p_0(\theta)$ so that $p(\theta|y) > 0$

2. Sample a candidate θ^* from a jumping (or candidate) distribution at time t: $\theta^* \sim J_t(\theta^* \mid \theta^{t-1})$which has to be symmetric

$$J_t(x \mid y) = J_t(y \mid x) \quad \text{for all} \quad x, y, t. \tag{22}$$

3. Calculate the density ratio

$$r = p(\theta^* \mid y)/p(\theta^{t-1} \mid y). \tag{23}$$

4. Accept the candidate with probability $min(r, 1)$

$$\theta^t = \begin{cases} \theta^* & \text{with prob.} \quad min(r,1) \\ \theta^{t-1} & \text{else.} \end{cases} \tag{24}$$

By cycling through step 1) to step 4) iterate for $t = 1, 2, ...$ until convergence. Note that if candidate draw is not accepted, then the sampler does not move: $\theta^{(t)} = \theta^{(t-1)}$. A good proposal is often difficult to obtain and in general the acceptance ratio should be as high as possible.

7 References

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