Bayesian causality measures for multiple ARCH models using marginal likelihoods

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Abstract

This paper extends the Geweke (1982, 1984) causality measures and suggests a causality decomposition procedure for Bayesian vector autoregressive processes (VAR) ased on marginal likelihoods. In a Bayesian context the causality measures can e interpreted as decompositions of the log Bayes factors. A further advantage is that these Bayesian causality measures can e extended to VAR-$VARCH$ and VAR-$VARCH$ 'in-mean' models. Furthermore, new causality decompositions with respect to the complexity of the $VARCH$ models are possi le. We demonstrate the approach y a financial example involving stock returns of the Dow-Jones, the DAX and the Nikkei indices.

Keywords: Bayesian VAR and VAR-$VARCH$-M models, Granger causality in multiple time series, Geweke feedback measures, marginal likelihoods, Bayes factors.
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1 Introduction

The (temporal) causality concept was introduced by Granger (1969) into econometrics and has triggered several controversial debates. A good survey of the early causality measures can be found in Pierce and Haugh (1977). Zellner (1978) and Leamer (1985) give a good historical perspective and many critical economic comments. Nevertheless, Wiener-Granger or temporal causality analyses are enjoying a remarkable popularity in econometrics and also in time series applications in other sciences. One reason is the convenient complexity reduction and the ability to condense information in complex multivariate time series systems. Geweke (1982, 1984) extended the concept to feedback measures in the time and frequency domain which can be decomposed and allow also to define conditional causality measures. Polasek (1994) showed how adjusted causality measures can be calculated from time series models which are measured by the information criteria AIC or BIC.

Surprisingly, no new developments have been made for causality measures in the last decade, despite many more multivariate time series techniques were developed in this period and no Bayesian contributions can be found. This paper aims to close this gap by introducing Bayesian (temporal) causality measures based on the marginal likelihood concept. From a Bayesian point of view the ratio of marginal likelihoods can be interpreted as Bayes factors testing different time series models. The equivalent of the Geweke's feedback measure decomposition is the decomposition of Bayes factors (i.e. the ratio of marginal likelihoods) or the likelihood ratio statistic (i.e. the ratio of maximized likelihood functions) for causal time series models.

We use the approach of Polasek and Ren (2000) for the calculation of marginal likelihoods based on the approach of Chib an Jeliazkov (1999) and informative priors which use the tightness prior as in Litterman (1986). Bayesian univariate AR (sometimes denoted as B-AR) and vector AR (B-VAR) time series models can be calculated by the normal-gamma or the normal-Wishart conjugate linear model and allows the computation of the marginal likelihoods in closed form. Using the Gibbs sampler and MCMC (Markov Chain Monte Carlo) methods (see Pelloni and Polasek 1997) we can extend the B-VAR model to B-$VARCH$ (autoregressive conditional heteroskedasticity) models and $VARCH$-M (vector $ARCH$-in-mean) models. Again by computing the exact marginal likelihood for Bayesian $VARCH$ models as e.g. in Chib and Jeliazkov (1999), we can extend the temporal causality to $VARCH$ models.

The plan of the paper is as follows. In section 1.1 to 1.3 we discuss the Granger causality concept and we define the conditional and unconditional Geweke or AR/VAR based causality measures and show how they re decomposed into the unidirectional and instantaneous causality parts. In section 3 we introduce Bayesian temporal causality measures and extend them to $ARCH$ based causality measures. Now we are faced with a wide spectrum of conditional causality measures and we have to explain the differences between them and we need some sophisticated notations. These efforts are needed to understand the double and two-way causality factorizations which are possible for nested and a non-nested ARCH processes. Section 4 extends the volatility based causality concept to $ARCH$-M based causality measures and propose a triple causality factorizations. Section 5 discusses financial market examples involving the daily returns of the Dow-Jones, the DAX, the Nikkei and the Hang Seng index. We show how the triple causality factorizations can be used to detect spurious and masked causality effects. A final section summarizes the merits of these extensions of causality measures for multiple volatile time series.

1.1 Temporal causality measures

Following Granger (1969) we define temporal causality in the following way. Let $\{I_t\}$ be the information set up to time t which includes at least a two time series x_t and y_t. Denote by $\{X_{(t)}\} = \{x_{t-i}, i = 1, 2, ...\}$ the past of the time series x_t and let $\sigma^2(Y|A)$ be the mean square error (MSE) if Y is explained with the information set A. Then we define

- X causes Y if $\sigma^2(y_t|I_{(t)})) < \sigma^2(y_t|I_{(t)}\backslash X_{(t)})$

  which means that the past of the time series x_t decreases the forecasting variance of y_t.
- X causes Y instantaneously if $\sigma^2(y_t|I_{(t)}, x_t) < \sigma^2(y_t|I_{(t)})$ which means that the current value of x_t improves the forecasting variance of y_t.

In this paper we will shift the focus of the causality from the predictive variance to the likelihood of the time series model. We define the temporal causality by the likelihood $l(y_t|I_{(t)})$ given the information set $I_{(t)} = (X_{(t)}, Y_{(t)})$:

- X causes Y if $l(y_t|I_{(t)}) > l(y_t|I_{(t)}\backslash X_{(t)})$. i.e. the model without the time series y_t gives a worse fit in terms of the likelihood values.

- X causes Y instantaneously if $l(y_t|I_{(t)}, x_t) > l(y_t|I_{(t)})$ i.e. the instantaneous observation x_t increases the likelihood of the time series model.

The two definitions are equivalent if the higher likelihood for a model leads to a smaller MSE. Instead of the likelihoods the use of information criteria AIC and BIC was proposed in Polasek (1994). The likelihood definitions of the temporal causality has the advantage that it can be used in classical and Bayesian inference. In the classical approach the parametric model is estimated by ML and the maximum likelihood values can be used to calculate and test the time series models:

$$l(y_t|I_{(t)}) = l(y_t|I_{(t)}, \hat{\theta})$$

where $\hat{\theta}$ is the maximum likelihood estimate of the model. For Bayesian inference we need a prior distribution $p(\theta)$ for the parameters and the marginal likelihoods are obtained by integrating the likelihood function over the prior distribution

$$l(y_t|I_{(t)}) = \int l(y_t|I_{(t)}, \theta)p(\theta)d\theta.$$

Many econometric program packages calculated the maximum likelihood values for the normal distribution. Marginal likelihoods for Bayesian models are more difficult to obtain because of a limited availability of Bayesian software packages. In the present study we will use the BASEL package of Polasek (2000).

1.2 AR/VAR based causality measures

In the following section we consider an autoregressive process of order p ($AR(p)$), defined as

$$y_t = \beta_0 + \beta^{'} y_{pt} + u_{1t}, \quad u_{1t} \sim N[0, \sigma_y^2], \tag{1}$$

and an $AR(p)X(p)$ is and $AR(p)$ process with the exogenous variables x up to lag p

$$y_t = \beta_0 + \beta_1^{'} y_{pt} + \beta_2^{'} x_{pt} + u_{2t}, \quad u_{2t} \sim N[0, \sigma_{y|x}^2] \tag{2}$$

with $y_{pt} = (y_{t-1}, \ldots, y_{t-p})^{'})$ and $x_{pt} = (x_{t-1}, \ldots, x_{t-p})^{'})$.
In similar way we define $AR_x X_{yz}$ processes or $AR_x X_y$ where x and y could also denote vector time series. σ_x^2 is the residual variance of AR_x process and $\sigma_{y|x}^2$ is the residual variance of the $AR_x X_y$ process. Σ_{xy} is the residual covariance matrix of the VAR process $(x_t, y_t)^{'}$.
The Geweke feedback (or temporal causality) measures are based on the logs of the residual variances of the AR processes for x_t and y_t in (1) and (2):

1. Unidirectional causality $x \rightarrow y$

$$F(x \rightarrow y) = ln \left(\frac{\sigma_y^2}{\sigma_{y|x}^2} \right).$$

2. Unidirectional causality $y \rightarrow x$

$$F(y \rightarrow x) = ln \left(\frac{\sigma_x^2}{\sigma_{x|y}^2} \right).$$

3. Instantaneous causality $x \cdot y$

$$F(x \cdot y) = ln \left(\frac{\sigma_{y|x}^2 \sigma_{x|y}^2}{|\Sigma_{xy}|} \right).$$

4. Linear dependence x, y

$$F(x, y) = ln \left(\frac{\sigma_y^2 \sigma_x^2}{|\Sigma_{xy}|} \right).$$

The idea of the Geweke causality measures is based on the ratio of improvement of the forecasting ability of stationary time series measured by the residual variances of AR processes. $F(x \rightarrow y)$ is roughly the percentage gain in the variances of the one-step ahead prediction if the variable x is added as exogenous variable to a simple AR process of the variable y. The unidirectional causality is zero, i.e. $F(x \rightarrow y) = 0$ only if there is no residual variance improvement if the variable x is added to the AR_y process. In the same way the unidirectional causality measure $F(y \rightarrow x)$ can be interpreted. The linear dependence measure is easily seen to be the maximum improvement of the forecasts if we compare two univariate AR processes for x and y. If we denote the joint residual covariance matrix by

$$\Sigma_{xy} = \begin{pmatrix} \sigma^2_{xx} & \sigma^2_{xy} \\ \sigma^2_{yx} & \sigma^2_{yy} \end{pmatrix}.$$

then we see that the linear dependence measure is only zero, i.e. $F(x \cdot y) = 0$ if the covariance between x and y is zero $\sigma^2_{xy} = 0$. In this case the determinant $|\Sigma_{xy}|$ is given by the product $\sigma^2_x \cdot \sigma^2_y$ and the bivariate VAR_{xy} system lacks any (temporal) causality since no forecasting improvement in terms of second moments can be found. Note that the linear dependence measure is a sum of 3 causality components

$$F(x,y) = F(x \rightarrow y) + F(y \rightarrow x) + F(x \cdot y).$$

Furthermore the instantaneous causality part can be explained as a remainder term: while the unidirectional contributions $F(x \rightarrow y)$ and $F(y \rightarrow x)$ indicate the strength of predictive performances over time, the contemporaneous part explains how much of the predictability is explained simultaneously at the same time period. If $F(x \cdot y) = 0$ then the time series are both explained by their own past and not contemporary. If $F(x \cdot y) \neq 0$ and $F(x \rightarrow y) = F(y \rightarrow x) = 0$ then we are confronted with a non (temporal) feedback situation: Both variable x and y are uncorrelated with their past and they are correlated in the same time period.

1.3 Conditional causality for an AR process

Let σ^2_x denote the residual variance of an AR process of the univariate time series $\{x_t\}$, $\sigma^2_{x|y}$ the residual variance of an $AR_x X_y$ process and by —Σ_{xy}— the generalized variance (the determinant of the covariance matrix) of the residuals of an VAR_{xy} process. Following Geweke (1984) we define the following conditional causality measures:

1. Conditional unidirectional causality $x \rightarrow y|z$

$$F(x \rightarrow y|z) = ln\left(\frac{\sigma^2_{y|z}}{\sigma^2_{y|xz}}\right).$$

2. Conditional unidirectional causality $y \rightarrow x|z$

$$F(y \rightarrow x|z) = ln\left(\frac{\sigma^2_{x|z}}{\sigma^2_{x|yz}}\right).$$

3. Conditional instantaneous causality $y \cdot x|z$

$$F(x \cdot y|z) = ln\left(\frac{\sigma^2_{x|yz}\sigma^2_{y|xz}}{|\Sigma_{xy|z}|}\right).$$

4. Conditional linear dependence $y, x|z$

$$F(x, y|z) = F(x \to y|z) + F(y \to x|z) + F(x \cdot y|z).$$

2 Bayesian temporal causality analysis

2.1 Bayes factors and marginal likelihood

When comparing any two models the Bayes factor (BF) can be calculated using the marginal likelihood concept. In general terms, letting y denote the relevant data set and θ_j be the appropriate vector of parameters under model M_j, the marginal likelihood can be defined as average or expected likelihood function, i.e.

$$f(y|M_j) = \int f(y|\theta_j, M_j)\, f(\theta_j|M_j)\, d\theta_j, \tag{3}$$

which depends on the model M_j and the prior distribution of the parameters θ_j of model M_j. If we denote by $f(y|M_1)$, the marginal likelihood for model 1, and by $f(y|M_2)$, the marginal likelihood for model 2, and if the two models are equally likely a priori, i.e. $P(M_1) = P(M_2) = 0.5$ we obtain the BF for M_2 versus M_1 by the ratio

$$BF_{21} = \frac{f(y|M_2)}{f(y|M_1)}. \tag{4}$$

Using the usual rules for computing odds, we find the posterior probabilities for models M_1 and M_2 by suitable normalisations. In particular, one may want to use the so-called $9 : 19 : 99$ rule for evaluating Bayes factors: $BF > 9$ indicate remarkable, $BF > 19$ are significant, and $BF > 99$ are highly significant different models.
Using logs we can transform this scheme to log-Bayes factors

$$\ln BF_{21} = \ln f(y|M_2) - \ln f(y|M_1). \tag{5}$$

One can use solely the differences of the log marginal likelihoods to judge the importance of models. If $\ln f(y|M_2)$ is the log-marginal likelihood of model 2 and $\ln f(y|M_1)$ is the log-marginal likelihood of model 1 then the $9 : 19 : 99$ cut-off points can be calculated for the log Bayes factor: $\ln 9 = 2.2$, $\ln 19 = 2.9$ and $\ln 99 = 4.6$ are the logs of the cut-off point. Roughly speaking this mean that we can use the numbers 2, 3 and 4 to judge the improvements in the model fit by looking only at the log marginal likelihoods.

2.2 Bayesian causality measures based on marginal likelihoods

Let x and y be 2 observed time series and let x_p and y_p be the past of the time series up to order p, then we define by $\tilde{R} = \tilde{R}_p$ the temporal causality measures based on p lags as the difference of two log marginal likelihoods. Now $AR_y(p)$ denotes the marginal likelihood of a $AR(p)$ process for y while $VAR_{xy}(p)$ denotes the marginal likelihood of a $VAR(p)$ process. In particular we define in dependence of the order p the following temporal causality measures:

1. Unidirectional causality $x \rightarrow y$

$$\tilde{R}(x \rightarrow y) = ln \left(\frac{AR_y(p)}{AR_y X_x(p)} \right) \tag{6}$$

2. Unidirectional causality $y \rightarrow x$

$$\tilde{R}(y \rightarrow x) = ln \left(\frac{AR_x(p)}{AR_x X_y(p)} \right)$$

3. Instantaneous causality $x \cdot y$

$$\tilde{R}(x \cdot y) = ln \left(\frac{AR_y X_x(p) \cdot AR_x X_y(p)}{VAR_{xy}(p)} \right)$$

4. Linear dependence (x, y)

$$\tilde{R}(x, y) = \tilde{R}(x \rightarrow y) + \tilde{R}(y \rightarrow x) + \tilde{R}(x \cdot y)$$

For notational convenience we will drop the dependence on p henceforth.

2.3 Conditional causality measures based on marginal likelihoods

In analogy to Geweke (1984) we define conditional causality measures for AR/VAR processes in the following way. Consider a 3-dimensional time series $\{x_t, y_t, z_t\}$, $t = 1, \ldots, T$ where the first two components are modeled by univariate AR processes with the exogenous variable z and the marginal likelihoods are denoted by $AR_x X_z$ or $AR_y X_z$. For a bivariate VAR model with the exogenous variable z the marginal likelihoods are denoted by $VAR_{xy} X_z$. For simplicity and notational reasons we drop now the explicit dependence of the marginal likelihood on p, i.e. $AR_x(p) = AR_x$

1. Conditional unidirectional causality $y \rightarrow x \mid z$

$$\tilde{R}(y \rightarrow x \mid z) = ln \left(\frac{AR_x X_z}{AR_x X_{yz}} \right) \tag{7}$$

2. Conditional unidirectional causality $x \to y \mid z$

$$\tilde{R}(x \to y \mid z) = ln \left(\frac{AR_y X_z}{AR_y X_{yz}} \right)$$

3. Conditional instantaneous causality $x \cdot y \mid z$

$$\tilde{R}(x \cdot y \mid z) \quad = \quad ln \left(\frac{AR_x X_{yz} \cdot AR_y X_{yz}}{VAR_{xy} X_z} \right)$$

4. Conditional linear dependence $(x, y) \mid z$

$$\tilde{R}((x,y) \mid z) = \tilde{R}(x \to y \mid z) + \tilde{R}(y \to x \mid z) + \tilde{R}(x \cdot y \mid z)$$

2.4 Extension of causality measures to sets of time series

Let $\mathbf{x} = (x_1, \ldots, x_s)$ and $\mathbf{y} = (y_1, \ldots, y_s)$ two sets of time series. Then we define in terms of the log marginal likelihoods the following vector causality measures:

1. Unidirectional causality $\mathbf{x} \to \mathbf{y}$

$$\tilde{R}(\mathbf{x} \to \mathbf{y}) = \ln \left(VAR_{\mathbf{y}} X_{\mathbf{x}}(p) \right) - \ln \left(VAR_{\mathbf{y}}(p) \right) \tag{8}$$

2. Unidirectional causality $\mathbf{y} \to \mathbf{x}$

$$\tilde{R}(\mathbf{y} \to \mathbf{x}) = \ln \left(VAR_{\mathbf{x}} X_{\mathbf{y}}(p) \right) - \ln \left(VAR_{\mathbf{x}}(p) \right)$$

3. Instantaneous causality $\mathbf{x} \cdot \mathbf{y}$

$$\tilde{R}(\mathbf{x} \cdot \mathbf{y}) = \ln \left(VAR_{\mathbf{y}} X_{\mathbf{x}}(p) \right) + \ln \left(VAR_{\mathbf{x}} X_{\mathbf{y}}(p) \right) - \ln \left(VAR_{\mathbf{xy}}(p) \right)$$

4. Linear dependence $(\mathbf{x}, \mathbf{y})$

$$\tilde{R}(\mathbf{x}, \mathbf{y}) = \tilde{R}(\mathbf{x} \to \mathbf{y}) + \tilde{R}(\mathbf{y} \to \mathbf{x}) + \tilde{R}(\mathbf{x} \cdot \mathbf{y})$$

3 $ARCH$ based causality measures

This section explores various AR-$ARCH$ based causality measures which extend the conditional causality concept of the previous section and that of Geweke (1984). We compute the causality measures from the marginal likelihood of Bayesian AR-$ARCH$ models and show that there exists a whole variety of different measures. Some of them are related by a simple decomposition of causality measures.
In extension of section 2 we define the AR-$ARCH$ process in the following way. A simple $ARCH_x(p)$ process without an AR component is defined as

$$y_t = \mu + u_t \quad or \quad y_t \sim N[\mu, h_t] \tag{9}$$

with the variance equation

$$h_t = \alpha_0 + \alpha^{'}\tilde{h}_{yt} \quad with \quad \tilde{h}_{yt} = (h_{t-1}, \ldots, h_{t-p})^{'}, \tag{10}$$

while the variance equation of a simple $GARCH(p,q)$ process is parameterized as

$$h_t = \alpha_0 + \alpha^{'}\tilde{h}_{xt} + \gamma^{'}\tilde{u}_{yt} \quad with \quad \tilde{u}_{yt} = (u_{t-1}^2, \ldots, u_{t-q}^2)'. \tag{11}$$

3.1 Conditional AR based causality measures

In this section we apply the conditional causality concept of Geweke (1984) to Bayesian AR-$ARCH$ processes. We consider the variable z in the AR and the $ARCH$ part and use the marginal likelihood as a measures of goodness of fit. An $AR_y ARCH_y X_x$ process is defined as an extension of (9) and (11) as

$$y_t \sim N[\mu_{yt}, h_{yt}],$$

where

$$\mu_{yt} = \beta_0 + \beta_1^{'}\tilde{y}_t \quad with \quad \tilde{y}_t = (y_{t-1}, \ldots, y_{t-p})^{'},$$

and

$$h_{yt} = \alpha_0 + \alpha_1^{'}\tilde{h}_{yt} + \alpha_2^{'}\tilde{h}_{xt}$$
$$\tilde{h}_{yt} = (h_{yt-1}, \ldots, h_{yt-p})', \quad \tilde{h}_{xt} = (h_{xt-1}, \ldots, h_{xt-q})',$$

where the exogenous variables of the ARCHX part of the model are taken from the ARCH model of the x_t process

$$x_t \sim N[\mu_{xt}, h_{xt}].$$

In a similar way ARX-$ARCHX$ processes can be defined and the conditional variances of the y_t process are used as exogenous variables. Now we can extend the AR based conditional causality measures (in VAR models) of section 2.3 to conditional AR based causality measure in $VARCH$ models in the following way:

1. Conditional causality $x \to y$ in AR-$ARCH$ models

$$\tilde{R}(x \to y \mid yARCH_{xy}) = ln\left(\frac{AR_y ARCH_y X_x}{AR_y X_x ARCH_y X_x}\right) \tag{12}$$

2. Conditional causality $y \to x$ in AR-$ARCH$ models

$$\tilde{R}(y \to x \mid xARCH_{yx}) = ln\left(\frac{AR_x ARCH_x X_y}{AR_x X_y ARCH_x X_y}\right).$$

3. Conditional instantaneous causality $x \cdot y$ in AR-$ARCH$ models

$$\tilde{R}(x \cdot y \mid xARCH_{yx}) \quad = \quad ln\left(\frac{AR_y X_y ARCH_y X_x \cdot AR_x X_y ARCH_x X_y}{VAR_{xy} VARCH_{xy}}\right)$$

4. Conditional linear dependence x,$y|z$ in AR-$ARCH$ models

$$\begin{aligned}\tilde{R}((x,y) \mid xARCH_{xy}) \quad &= \quad \tilde{R}(x \to y|yARCH_{xy}) + \tilde{R}(y \to x|xARCH_{yx}) + \\ &+\tilde{R}(x \cdot y|yARCH_{xy})\end{aligned}$$

3.2 AR causality measures in AR-$ARCH$ models

In extension of the AR and VAR based causality measures of the previous section we define conditional $ARCH$ based causality measures using marginal likelihoods of $ARCH$ models. We look at the causal x effects in the mean and the variance equation at the same time. This is accomplished by comparing a simple (univariate) AR-$ARCH$ model with an ARX-$ARCHX$ model: the x_t process enters in lags as an exogenous variable in the mean equation and by lagged conditional variances in the variance equation. For simplicity we assume the same lag length p for both equations.

1. Unidirectional causality $x \to y$ in AR-$ARCH$ models

$$\tilde{R}(x \to y \mid ARCH) = \ln\left(\frac{AR_y ARCH_y}{AR_y X_x ARCH_y X_x}\right) \tag{13}$$

2. Unidirectional causality $y \to x$ in AR-$ARCH$ models

$$\tilde{R}(y \to x \mid ARCH) = ln\left(\frac{AR_x ARCH_x}{AR_x X_y ARCH_x X_y}\right).$$

3. Instantaneous causality $x \cdot y$ in AR-$ARCH$ models

$$\tilde{R}_p(x \cdot y \mid ARCH) = \ln\left(\frac{AR_y X_y ARCH_y X_x \cdot AR_x X_y ARCH_x X_y}{VAR_{xy} VARCH_{xy}}\right)$$

4. Linear dependence (x,y) in AR-$ARCH$ models

$$\begin{aligned}\tilde{R}((x,y) \mid ARCH) &= \tilde{R}(x \to y \mid ARCH) + \tilde{R}_p(y \to x \mid ARCH) + \\ &\quad + \tilde{R}(x \cdot y \mid ARCH)\end{aligned}$$

3.3 x-in-$ARCH$ causality measures for pure $ARCH$ models

In this section we define special 'pure volatility based' or 'x-in- $ARCH$' causality measures in processes without an AR component. These causality measures can be viewed as pure volatility based causality measures. We denote them by $\tilde{V}_p$ to highlight the volatility nature and the dependence on the order of the process p.

1. Unidirectional x-in-$ARCH$ causality measure $h_x \to h_y$

$$\tilde{V}(x \to y) = ln\left(\frac{ARCH_y}{ARCH_y X_x}\right) \tag{14}$$

2. Unidirectional x-in-$ARCH$ causality measure $h_y \to h_x$

$$\tilde{V}(y \to x) = ln\left(\frac{ARCH_x}{ARCH_x X_y}\right)$$

3. Instantaneous x-in-$ARCH$ causality measure $h_y \cdot h_x$

$$\tilde{V}(x \cdot y) \quad = \quad ln\left(\frac{ARCH_yX_x \cdot ARCH_xX_y}{VARCH_{xy}}\right)$$

4. Linear dependence in x-in-$ARCH$ models

$$\tilde{V}(y,x) \quad = \quad \tilde{V}(x \rightarrow y) + \tilde{V}(y \rightarrow x) + \tilde{V}(y \cdot x).$$

The conditional variances are calculated as

$$h_{xt} \quad = \quad \alpha_0 + \sum_i \alpha_i h_{x,t-1} + \sum_i \gamma_i u_{t-i}^2.$$

Note: For practical computations we found the following 'volatility switching' principle as useful. Compute the marginal likelihood by using as the exogenous volatility variable the current volatility component of the MCMC run:

- The exogenous regressors in the $ARCH_yX_x$ component in the denominator of $\tilde{V}(x \cdot y)$ in is taken from the simple $ARCH_x$ model which appears in the nominator in 2.
- The exogenous regressor of $ARCH_xX_y$ is computed in similar way from the $ARCH_y$ model in 1.

3.4 Conditional $ARCH$ based causality measures

Alternatively we define the following conditional $ARCH$ based causality measures. The term $ARCH$ based refres to the fact that the ARCH component is considered as source of causlity in the ARX model where the ARCHX component is added. The notation $ARCH_xX_y \rightarrow y|xy$ means that the autoregressive component in the mean equation is held constant for all causality measures while only the $ARCHX$ component is allowed to improve the fit of the model.

1. Conditional $ARCH$ based causality measure $x \rightarrow y$

$$\tilde{R}(xARCH_{xy} \rightarrow y|xy) = ln\left(\frac{AR_yX_x}{AR_yX_xARCH_yX_x}\right) \tag{15}$$

2. Conditional $ARCH$ based causality measures $y \rightarrow x$

$$\tilde{R}(yARCH_{xy} \rightarrow x|xy) = ln\left(\frac{AR_xX_y}{AR_xX_yARCH_xX_y}\right)$$

3. Conditional $ARCH$ based instantaneous causality $x \cdot y$

$$\tilde{R}(xARCH_y \cdot yARCH_x|xy) = ln\left(\frac{AR_yX_xARCH_yX_x \cdot AR_xX_yARCH_xX_y)}{VAR_{xy}VARCH_{xy}}\right)$$

4. Conditional $ARCH$ based conditional dependence x, y

$$\begin{aligned}\tilde{R}(xARCH_y, yARCH_x|xy) &= \tilde{R}(ARCH_{xy} \to y|xy) + \\ &+\tilde{R}(ARCH_{xy} \to x|xy) + \\ &+\tilde{R}(ARCH_xX_y \cdot ARCH_yX_x|xy).\end{aligned}$$

3.5 AR-$ARCH$ based causality measures

This section defines (unconditional) causality measures based on marginal likelihoods of AR-$ARCHX$ models. There is only a slight difference in the definition between the 'conditional $ARCH$ based causalities' and the 'AR-$ARCH$ based causalities'. While the conditional measures have an intuitive appeal, the 'AR-$ARCH$ based causalities' will appear in the causality factorizations. Therefore it is necessary to clarify the difference between these two sets of causality measures. We use also the shorter notation $xARCH_{xy}$ for the process $AR_xARCH_xX_y$.

1. Unidirectional AR-$ARCH$ based causality $x \to y$

$$\tilde{R}(xARCH_{xy} \to y) = \ln\left(\frac{AR_y}{AR_yX_xARCH_yX_x}\right) \qquad (16)$$

2. Unidirectional AR-$ARCH$ based causality $y \to x$

$$\tilde{R}(yARCH_{xy} \to x) = \ln\left(\frac{AR_x}{AR_xX_yARCH_xX_y}\right)$$

3. Instantaneous AR-$ARCH$ based causality $(y \cdot x)$

$$\tilde{R}(xyARCH_{x \cdot y}) = \ln\left(\frac{AR_yX_xARCH_yX_x \cdot AR_xX_yARCH_xX_y}{VAR_{xy}VARCH_{xy}}\right)$$

4. Total AR-$ARCH$ based linear dependence (x, y)

$$\begin{aligned}\tilde{R}(xyARCH_{x,y}) &= \tilde{R}(xARCH_{xy} \to y) + \tilde{R}(yARCH_{xy} \to x) + \\ &+\tilde{R}(xyARCH_{x \cdot y}).\end{aligned}$$

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basedausalit

entarysi4e6dvxIfij8FFESFAAXAjAOqWI8WWAXOAXW8WAWOAAAjAtqAXa6dvord

1. Unidirectional AR-$ARCH$ based causality $x \to y$ factorization

$$\tilde{R}(xARCH_{xy} \to y) = \tilde{R}(x \to y) + \tilde{R}(xARCH_y \to y|xy)$$

2. Unidirectional AR-$ARCH$ based causality $y \to x$ factorization

$$\tilde{R}(yARCH_{xy} \to x) \quad = \quad \tilde{R}(y \to x) + \tilde{R}(yARCH_x \to x|xy)$$

3. Instantaneous AR-$ARCH$ based causality $(x \cdot y)$ factorization

$$\tilde{R}(xyARCH_{x \cdot y}) \quad = \quad \tilde{R}(x \cdot y) + \tilde{R}(VARCH_{xy} \to VAR_{xy})$$

4. Total AR-$ARCH$ based linear dependence (x, y) factorization

$$\tilde{R}(xyARCH_{x,y}) \quad = \quad \tilde{R}(x, y) + \tilde{R}(VARCH_{xy} \to VAR_{xy}).$$

Proof: We just show how the unidirectional causality $x \to y$ can be decomposed. The first factorization is obtained by

$$\ln\left(\frac{AR_y}{AR_yX_xARCH_yX_x}\right) \quad = \quad \ln\left(\frac{AR_y}{AR_yX_x}\right) + \ln\left(\frac{AR_yX_x}{AR_yX_xARCH_yX_x}\right)$$

or using the AR-$ARCH$ based causality notation of section 3.5

$$\tilde{R}(xARCH_{xy} \to y) = \tilde{R}(x \to y) + \tilde{R}(xARCH_{xy} \to y|xy).$$

The other causality measures are similarly decomposed. Note that the factorization of the instantaneous part and the linear dependence follow a different pattern. The second factor of the decomposition is in both cases the same and can be interpreted as a multivariate causality measures:

$$\tilde{R}(VARCH_{xy} \to VAR_{xy}) = \ln\left(\frac{VAR_{xy}}{VAR_{xy}VARCH_{xy}}\right).$$

The nested $ARCH$ factorization starts with the AR processes and explains the total causality potential of an ARX-$ARCHX$ model by the Geweke (1982) unconditional causality measures plus a remainder term due to the $ARCH$ component. The second way to factorize causality measures is to start with the $ARCH$ based causality measures and add to this the AR based causality measures conditional on the $ARCH$ part. This is shown in the next theorem.

Theorem 3.2 Double causality factorization (nested and non-nested $ARCH$)

We show that the AR-$ARCH_{xy}$ based causality of section 3.5 can be decomposed into an $ARCH_{xy}$ based causality plus and AR based conditional causality measure (defined in section 3.2). There are two possible ways to factorize the AR-$ARCHX$ causalities measure (section 3.7) or the non-nested $ARCH$ causalities measure (section 3.8) which can be used only in special cases.

1. Unidirectional AR-$ARCH$ based causality $x \rightarrow y$ factorization

$$\tilde{R}(xARCH_{xy} \rightarrow y) = \begin{cases} \tilde{R}(ARCH_x \rightarrow y) + \tilde{R}(x \rightarrow y|ARCH_{xy}) \\ \tilde{R}(yARCH_{xy} \rightarrow y) + \tilde{R}(x \rightarrow y|yARCH_{xy}). \end{cases}$$

2. Unidirectional AR-$ARCH$ based causality $y \rightarrow x$ factorization

$$\tilde{R}(yARCH_y \rightarrow x) = \begin{cases} \tilde{R}(ARCH_{yx} \rightarrow x) + \tilde{R}(y \rightarrow x|ARCH_{yx}) \\ \tilde{R}(xARCH_{yx} \rightarrow x) + \tilde{R}(y \rightarrow x|xARCH_{yx}) \end{cases}$$

3. Instantaneous AR-$ARCH$ based causality $(x \cdot y)$ factorization

$$\tilde{R}(xyARCH_{x \cdot y}) = \tilde{R}(ARCH_x \cdot ARCH_y) + \tilde{R}(VAR_{xy} \rightarrow VARCH_{xy}) \quad (17)$$

with $\tilde{R}(ARCH_x \cdot ARCH_y)$ being the instantaneous causality of the $ARCH$ process and $\tilde{R}(VAR_{xy} \rightarrow VARCH_{xy})$ is the remainder term.

4. Total AR-$ARCH$ based linear dependence (x, y) factorization

$$\tilde{R}(xyARCH_{x,y}) = \tilde{R}(x,y) + \tilde{R}(VARCH_{xy} \rightarrow VAR_{xy}),$$

where $\tilde{R}(x,y)$ is the measure of linear dependence for the AR and VAR processes and $\tilde{R}(VARCH_{xy} \rightarrow VAR_{xy})$ is remainder term. Note that the remainder term in 3. and 4. have different causality directions.

Proof: The instantaneous causality for non-nested $ARCH$ version is decomposed as

$$\begin{aligned} \tilde{R}(xyARCH_{x.y}) &= \ln\left(\frac{ARCH_yX_x \cdot ARCH_xX_y}{VARCH_{xy}} \cdot \frac{VARCH_{xy}}{VAR_{xy}VARCH_{xy}}\right) \\ &= \tilde{R}(ARCH_{yx} \cdot ARCH_{xy}) + \tilde{R}(VAR_{xy} \rightarrow VARCH_{xy}). \end{aligned}$$

Alternatively, the nested $ARCH$ version for the instantaneous decomposition is

$$\begin{aligned} \tilde{R}(xyARCH_{x.y}) &= \ln\left(\frac{AR_yARCH_yX_x \cdot AR_xARCH_xX_y}{VAR_{xy}} \cdot \frac{VAR_{xy}}{VAR_{xy}VARCH_{xy}}\right) \\ &= \tilde{R}(yARCH_x \cdot xARCH_y) + \tilde{R}(VARCH_{xy} \rightarrow VAR_{xy}) \end{aligned}$$

with

$$\tilde{R}(VAR_{xy} \rightarrow VARCH_{xy}) = \ln\left(\frac{VARCH_{xy}}{VAR_{xy}VARCH_{xy}}\right)$$

and

$$\tilde{R}(VARCH_{xy} \rightarrow VAR_{xy}) = \ln\left(\frac{VAR_{xy}}{VAR_{xy}VARCH_{xy}}\right).$$

Both $ARCH$ factorization uses the $ARCH_yX_x$ based causality part first and the remainder term is the conditional AR part which is left.

This decomposition can be calculated for different lag orders of the $AR(k)X(k)$-$ARCH(q)X(q)$ or $AR(k)X(k)$-$GARCH(p,q)X(q)$ processes. We suggest to select the model with the largest linear dependence measure. This means that we use for the causality decomposition the process where the information gain between the univariate processes and the multivariate process is maximal.
The second decomposition is the non-nested causality factorization

$$\ln\left(\frac{AR_y}{AR_yX_xARCH_yX_x}\right) = \ln\left(\frac{AR_y}{ARCH_yX_x}\right) + \ln\left(\frac{ARCH_yX_x}{AR_yX_xARCH_yX_x}\right),$$

or

$$\tilde{R}(xARCHX \to y) = \tilde{R}(ARCH_{xy} \to y) + \tilde{R}(x \to y|ARCH_{xy}).$$

The nested $ARCH$ factorization is given by

$$\ln\left(\frac{AR_y}{AR_yX_xARCH_yX_x}\right) = \ln\left(\frac{AR_y}{AR_yARCH_yX_x}\right) + \ln\left(\frac{AR_yARCH_yX_x}{AR_yX_xARCH_yX_x}\right),$$

or

$$\tilde{R}(xyARCH_{xy} \to y) = \tilde{R}(yARCH_{xy} \to y) + \tilde{R}(x \to y|yARCH_{xy}).$$

In order to see the difference between these two factorizations we have to define the sets of 'nested' and the 'non-nested' causality measures in section 3.7 and 3.8, respectively.

3.7 Nested $ARCH$ causality measures

The nested causality measures compares AR models with $AR-ARCH$ models, i.e. we add to the simple AR model an $ARCH$ component. Thus, in contrast to section 3.4 and 3.5 the nested causality measures are simple ratios of AR and AR-$ARCH$ processes.

1. Nested $ARCH$ based causality $x \to y$

$$\tilde{R}(xARCH_{yx} \to y) = \left(\frac{AR_y}{AR_yARCH_yX_x}\right).$$

2. Nested $ARCH$ based causality $y \to x$

$$\tilde{R}(yARCH_{xy} \to x) = \left(\frac{AR_x}{AR_xARCH_xX_y}\right).$$

3. Nested $ARCH$ based instantaneous causality

$$\tilde{R}(xyARCH_{x\cdot y}) = \left(\frac{AR_xARCH_xX_y \cdot AR_yARCH_yX_x}{VARCH_{xy}}\right).$$

4. Nested $ARCH$ based linear dependence

$$\tilde{R}(xyARCH_{x,y}) = \tilde{R}(yARCH_{yx} \to y) + \tilde{R}(xARCH_{xy} \to x) + \tilde{R}(xyARCH_{x\cdot y}).$$

3.8 The non-nested $ARCH$ causality measures

In models where the $ARCH$ component dominates it might be useful to consider non-nested $ARCH$ causality measures. The non-nested $ARCH$ causality measures are only useful if the $ARCHX$ fit is better than the AR fit, because then the ratio of AR to $ARCHX$ likelihoods is less than 1. If this condition is not met, then alternatively the nested $ARCH$ causality measures have to be used, in order to make the likelihood ratio greater than 1. Unfortunately, the factorization with nested $ARCH$ causalities might lead to smaller causality differences in applications.

1. Non-nested $ARCH$-*based* causality $x \to y$

$$\tilde{R}(ARCH_{yx} \to y) = \left(\frac{AR_y}{ARCH_y X_x} \right).$$

2. Non-nested $ARCH$-*based* causality $y \to x$

$$\tilde{R}(ARCH_{xy} \to x) = \left(\frac{AR_x}{ARCH_x X_y} \right).$$

3. Non-nested $ARCH$-*based* instantaneous causality $(ARCH_{x \cdot y})$

$$\tilde{R}(ARCH_{x \cdot y}) = \left(\frac{ARCH_x X_y \cdot ARCH_y X_x}{VARCH_{xy}} \right).$$

4. Non-nested $ARCH$-*based* linear dependence (y, x)

$$\tilde{R}(ARCH_{x,y}) = \tilde{R}(ARCH_{yx} \to y) + \tilde{R}(ARCH_{xy} \to x) + \tilde{R}(ARCH_{x \cdot y}).$$

4 Causality in $VARCH$-M models

This section extends the causality measures of the previous section where we have considered the class of univariate and multivariate ARX-$ARCHX$ models to the class of ARX-$ARCHX$-M models. Unfortunately, more complex models require more complicated notations and therefore we will outline only the important new features. The estimation of $ARCH$-in-mean models is quite demanding and therefore it will be interesting to see how important the new extensions to the $ARCH$ models are. Again we will quantifiy the model extensions by appropriate causality measures and a triple factoristaion where the third part will show the weight of the $ARCH$-in-mean component in terms of marginal likelihoods.

4.1 AR-$ARCH$-M based causality measures

This section defines (unconditional) causality measures based on marginal likelihoods of AR-$ARCH$-M models. Again, the 'AR-$ARCH$-M based causalities' will appear in the causality factorizations. Therefore it is necessary to clarify the difference between these two sets of causality measures. We use also the shorter notation $xARCH_{xy}$-M for the process $AR_xX_yARCH_xX_y$-M.

1. Unidirectional AR-$ARCH$-M based causality $x \to y$

$$\tilde{R}(xARCH_{xy}\text{-}M \to y) \quad = \quad \ln\left(\frac{AR_y}{AR_yX_xARCH_yX_x\text{-}M}\right)$$

2. Unidirectional AR-$ARCH$-M based causality $y \to x$

$$\tilde{R}(yARCH_{xy}\text{-}M \to x) \quad = \quad \ln\left(\frac{AR_x}{AR_xX_yARCH_xX_y\text{-}M}\right)$$

3. Instantaneous AR-$ARCH$-M based causality $(y \cdot x)$

$$\tilde{R}(xyARCH_{x\cdot y}\text{-}M) \quad = \quad \ln\left(\frac{AR_yX_xARCH_yX_x\text{-}M \cdot AR_xX_yARCH_xX_y\text{-}M}{VAR_{xy}VARCH_{xy}}\right)$$

4. Total AR-$ARCH$-M based linear dependence (x, y)

$$\tilde{R}(xyARCH_{x,y}\text{-}M) = \tilde{R}(xARCH_{x,y}\text{-}M \to y) + \tilde{R}(yARCH_{x,y}\text{-}M \to x) + \\ + \tilde{R}(xyARCH_{x\cdot y}\text{-}M).$$

4.2 Conditional x-in-$ARCHM$ causality measures

For the class of AR-$ARCH$-M models we can calculate AR based causality measures conditional on the $ARCH_{xy}M$ component in the following way:

1. Unidirectional x-in-$ARCHM$ causality measure $x \to y$

$$\tilde{R}_p(x \to y|ARCH_{yx}M) = \ln\left(\frac{AR_yARCH_yM_y}{AR_yX_xARCH_yX_xM_{xy}}\right) \tag{18}$$

2. Unidirectional y-in-$ARCHM$ causality measure $y \to x$

$$\tilde{R}(y \to x|ARCH_{xy}M) = \ln\left(\frac{AR_xARCH_xM_x}{AR_xX_yARCH_xX_yM_{xy}}\right)$$

3. Instantaneous xy-in-$ARCHM$ causality measure $y \cdot x$

$$\tilde{R}(y \cdot x|ARCH_{xy}M) = \ln\left(\frac{AR_yX_xARCH_yX_xM_{xy} \cdot AR_xX_yARCH_xX_yM_{xy}}{VAR_{xy}VARCH_{xy}M_{xy}}\right)$$

4. Total xy-in-$ARCHM$ linear dependence (y, x)

$$\tilde{R}(x, y) = \tilde{R}(x \to y) + \tilde{R}(y \to x) + \tilde{R}(y \cdot x)$$

Note: For the calculation of the exogenous x-part of the unidirectional causality measures in 1. and 2. use the volatility switching technique: for the 'volatility'-regressors which go into the M_x component we use the variance estimate of the current cycle of the MCMC algorithm.

4.3 Triple $ARCHM$ factorization

In extension of the previous section we propose the following triple causality factorization. This triple factorization leads to an 3^{rd} component by taking the effect of the $ARCH$-in-mean component separately into account. As in Theorem 3.1 and 3.2 we distinguish between an AR-*start* and an $ARCH$-*start* factorization. Both factorization are listed in Theorem 4.1, but the second factorization only exists if the marginal likelihoods satisfy $\ln AR < \ln ARCHX$.

Theorem 4.1 Triple causality factorization

1. Unidirectional $xARCHM$ based causality $x \rightarrow y$

$$\tilde{R}(xARCH_{xy}M \rightarrow y) = \begin{cases} \tilde{R}(x \rightarrow y) + \tilde{R}(xARCH_{xy} \rightarrow y|x) + \\ +\tilde{R}(xM_{xy} \rightarrow y|yxARCH_{yx}) \\ \tilde{R}(ARCH_{xy} \rightarrow y) + \tilde{R}(x \rightarrow y|ARCH_{xy}) + \\ +\tilde{R}(xM_{xy} \rightarrow y|yxARCH_{yx}). \end{cases} \quad (19)$$

or

$$\tilde{R}(xARCH_{xy}M \rightarrow y) = \tilde{R}(ARCH_{xy} \rightarrow y) + \tilde{R}(x \rightarrow y|ARCH_{xy}).$$

2. Unidirectional $xARCHM$ based causality $y \rightarrow x$

$$\tilde{R}(yARCH_{xy}M \rightarrow x) = \begin{cases} \tilde{R}(y \rightarrow x) + \tilde{R}(yARCH_{xy}M_x \rightarrow x|y) + \\ +\tilde{R}(yM_{xy} \rightarrow x|xyARCH_y) \\ \tilde{R}(ARCH_{xy} \rightarrow x) + \tilde{R}(y \rightarrow x|ARCH_{xy}) + \\ +\tilde{R}(yM_{xy} \rightarrow x|xyARCH_{xy}), \end{cases}$$

3. Instantaneous $xARCHM$ based causality $(x \cdot y)$

$$\tilde{R}(x \cdot y \cdot ARCH_{xy}M) = \begin{cases} \tilde{R}(x \cdot y) + \tilde{R}(xy \cdot VARCH) + Rest1 \\ \tilde{R}(xARCH_x \cdot yARCH_y) + \tilde{R}(x \cdot y|ARCH_{xy}) + Rest2 \end{cases}$$

Proof: For the unidirectional parts we show only how the $ARCH$-start factorization in the second line is obtained:

$$\ln\left(\frac{AR_y}{AR_yX_xARCH_yX_xM}\right) = \ln\left(\frac{AR_y}{ARCH_yX_x}\right) + \ln\left(\frac{ARCH_yX_x}{AR_yX_xARCH_yX_x}\right) + \\ + \ln\left(\frac{AR_yX_xARCH_yX_x}{AR_yX_xARCH_yX_xM}\right).$$

For the AR factorization simply replace the $ARCH_yX_x$ model by an AR_yX_x model while the 3^{rd} term stays the same.
For the instantaneous part we decompose the ARX-$ARCHXM$ into the ARX-$ARCHX$ instantaneous component plus the rest. Then we use the ARX-$ARCHX$ decomposition as before in section 3.6, i.e.

$$Rest1 = \tilde{R}(xy \cdot VARCHM) - \tilde{R}(xy \cdot VARCHX).$$

Therefore we find

$$\tilde{R}(x \cdot y \cdot ARCHXM) \quad = \quad \tilde{R}(x \cdot y) + \tilde{R}(xy \cdot VARCH) + Rest1.$$

In the same way we factorize the $ARCH$ rooted instantaneous causality:

$$Rest2 = \tilde{R}(x \cdot y | VARCH_{xy} M) + \tilde{R}(x \cdot y | ARCH_{xy})$$

where the second term is given in section 3.6.

4.4 Conditional $ARCHM$ causalities

Define as $AA_{yx} = AR_y X_x ARCH_y X_x$ and $AA_{xy} = AR_x X_y ARCH_x X_y$. Then we can define for the *ARCH-in-mean* component the following conditional causality measures:

1. Unidirectional conditional $ARCHXM$ causality $xM_{xy} \to y$

$$\tilde{R}(xM_{xy} \to y | yxARCH_{yx}) = \ln \left(\frac{AA_{yx}}{AA_{yx} M_{xy}} \right).$$

2. Unidirectional conditional $ARCHXM$ causality $yM_{xy} \to x$

$$\tilde{R}(yM_{xy} \to x | xyARCH_{xy}) = \ln \left(\frac{AA_{xy}}{AA_{xy} M_{xy}} \right).$$

3. Instantaneous conditional $ARCHXM$ causality (y, x)

$$\tilde{R}(xM_{xy} \cdot yM_{xy} | xyARCH_{xy}) = \ln \left(\frac{AA_{yx} M_{xy} \cdot AA_{xy} M_{xy}}{VAR_{xy} VARCH_{xy} M_{xy}} \right).$$

4. Linear dependence for $ARCHXM$ causality models (y, x)

$$\tilde{R}(xM_{xy}, yM_{xy} | xyARCH_{xy}) = \ln \left(\frac{AA_{yx} \cdot AA_{xy}}{VAR_{xy} VARCH_{xy} M_{xy}} \right).$$

5 Examples: Daily stock market returns

In this section we apply the previous causality measures for daily stock returns in the period from June 21, 1996 to June 22, 1998 which were analyzed before in Polasek and Ren (1998). Table 1 shows the conditional causality for an AR process. Table 2, 3 and 4 show the log marginal likelihood for the VAR-$VARCHX$-M family. Table 5 shows the simple AR/VAR based causality measures based on marginal likelihoods with informative tightness priors. The hyper-parameters of the prior distribution we choose as mean $\beta_* = (0, \ldots, 0)$, and as prior precision $H_*^{-1} = diag(\varepsilon, 1, 2, \ldots, p)$ with $\varepsilon = 0.001$, $s_*^2 = var(y)/2$ and $n_* = 1$. This decomposition can be calculated for different orders of the

$AR(k)X(k)ARCH(p)X(q)$ process. We suggest selecting the model with the largest linear dependence measure. This means that we use for the causality decomposition (and factorization) the process where the information gain between the univariate and the multivariate process is maximal.
Table 2 and 3 list the marginal likelihoods for the model selection of the DAX and the Dow-Jones returns for various order of the ARX-$ARCHX$ processes. The model $AR(2)X(2)ARCH(1)X(1)$ is marked with a star (*), because it is the best in the class of ARX-$ARCHX$ processes. Concerning the causality decomposition analysis we see that the linear dependence measure for the $AR(2)X(2)ARCH(2)X(2)$ process yields the largest information gain.
In the last column of table 2 and 3 we have listed the asymmetric '$aAR_yX_xARCH_xa$' model which contains an additional dummy variable for negative residuals. From the lower marginal likelihood values we see that this asymmetric extension of the AR-$ARCH$ model is inferior in comparison with the AR-$ARCH$-M model. Therefore we have not further investigated this class of models.

5.1 Triple causality factorization

Table 6 and 7 show the triple $xARCH$-M factorization for the best causality model of the $VARX$-$ARCHX$-M class. The systematic displays in Figures 1 and 2 will be useful to understand the various causality decompositions. Table 6 contains the factorization which starts with the AR-causality component first, while Table 7 shows the factorization which starts with the $ARCH$ causality component first. Each table consists of four row blocks and 4 column blocks. We start explaining the details of the first row block in Table 6.
The first row in Table 6 shows the marginal likelihoods for the unidirectional causalities $R(x \to y)$, $R(y \to x)$ and the instantaneous causality $R(x \cdot y)$ which adds up to the measure of linear dependence $R(x, y)$. Now y and x are assigend the univariate total and the multivariate sectoral time series of the employment series, respectively.
The second row in the first row block is called 'row%' and is is the relative decomposition of the overall $VARX$-$ARCHX$-M causality measures. The number 1 in the last column acts as an indicator that the 3 components add up to 1. The third row is called 'column%' and consists only of ones. This is a label for the causality measures which are decomposed for each column. The second row block of causality measures consists of the usual Geweke causality measures based on the AR and VAR processes. The relative causality contributions are shown in the second row with the label 'row%'. The third row has only 2 entries which are found in the instantaneous causality column and the last column of linear dependence. The 4th row is the column% decomposition and will be explained after the last row block.
The 3rd row block is the family of conditional causality measures $R(xARCH_{xy} \to y|x)$ where we condition on the AR causality part of the first row block. Again the precise definitions of the causality measures can be found in the overview of Figure 1.
In the second row we find the relative decomposition of these conditional causal-

ity measures. The 3rd and the 4th row is reserved for the column-wise causality decomposition.
The 4th row block is the family of conditional causality measures $R(xM_x \rightarrow y|xyARCH_{xy})$ where we condition now on the AR and the $ARCH$ causality part of the first and second row block. (Therefore the conditioning event is abbreviated with $'xyARCH'_{xy}$). Again the exact definitions of these causality measures can be found in the overview of Figure 1. In the second row we find the relative decomposition of these conditional causality measures. The third row belongs to the vertical log marginal likelihood decomposition. Now let us explain the 3rd and the 4th column in detail. The unconditional and the conditional causality measures only factorize and sum row-wise to 1 in relative terms, because that is how they are defined. Unfortunately these measures only add up for the first 2 columns, the unidirectional causality measures. They don't add up to 1 for the instantaneous causality part $R(x \cdot y)$ and the measure of linear dependence $R(x, y)$.
Therefore we propose to use another factorization for these 2 terms. For the second row block this term is just the instantaneous causality part $R(x \cdot y)$ and the measure of linear dependence $R(x, y)$ of the original Geweke causality measures. The 3rd row block is just the log ratio of the improvement of the VAR fit to the VAR-$VARCH$ fit, and the values are identical for the pre-last and the last column. The 4th row block is now the log ratio of the improvement of the VAR-$VARCH$ fit to the VAR-$VARCH$-M fit, and it is again the same for the pre-last and the last column. Now the 4th row of the 4th row block displays the relative decomposition of the column-wise factorization. While the unidirectional causality columns sum up to 1 by their original definitions, the last two columns are not adding up to 1 as can be easily seen from the tables. Therefore we have introduced the additional decomposition for the last 2 columns to complete the picture of a row-wise and column-wise decomposition of causality measures. The column-wise decomposition just shows how important are the 3 parts of a VAR-$VARCH$-M model in relative terms for the 4 components of the causality measures: the AR part, the $ARCH$ part and the $ARCH$-M part.

5.2 Spurious and masked causality

Conditional causality measures can be used to check for spurious and masked (hidden) causality. We speak of spurious causality if the unconditional causality is not zero $R(x, y) \neq 0$, but the conditional causality measure is zero: $R(x, y|z) = 0$. Masked (or suppressed) causality is just the opposite phenomenon: the unconditional causality is zero $R(x, y) = 0$, but the conditional causality measure is not zero: $R(x, y|z) \neq 0$.
The two possible causality factorizations in Tables 6 and 7 show that the pure AR part is rather small while all causality measures which involve $ARCH$ and $ARCHX$ influences, i.e. exogenous variables in the $ARCH$ equation, explain the causality measures to the highest extent, sometimes over 90%. Comparing these results with conditional causalities will show if the high instantaneous causality is spurious and if the low unidirectional causalities are masked.

Note that Table 7 contains the conditional causality measures $R(x \to y|ARCH_{xy})$ where the causality flow from x to y and also in the other direction is quantified when the effects of the $ARCH$ component is held constant. These conditional measures can be compared with the pure AR based causality measures $R(x \to y)$ which can be found in the second row block of Table 6. These AR based causality measures have a very high instantaneous causality component (90%) and only about 5% for each of the two unidirectional measures. Now we can compare these unconditional measures with the conditional causality measures in the 3rd row block of Table 7: We see that the instantaneous part drops to 35% in the relative decomposition and the other two unidirectional measures are about the same size, but certainly larger than the pure AR measures are suggesting. This shows that the AR causality flows are masked by the $ARCH$ component which seem to absorb much of the causality component in the AR part. Note that the 4th row block in Tables 6 and 7 show conditional causality measures of a different kind which is the volatility feedback on the returns, when both, the AR and the ARCH model influence is hold constant. The 20.8% number in the second column shows that the Dow-Jones volatility is a little bit more important for the DAX returns than the the DAX volatility for the Dow-Jones returns (15.7%). These analysis demonstrate that the triple causality decompositions for *AR-ARCH-M* causality measures are important tools for detecting potential spurious and masked causality relationships.
Clearly, other variables could have been responsible for spurious and masked causality effects as well. The first two panels of Table 5 compare the unconditional and conditional Geweke (or pure AR) causality measures. The efficient market hypothesis in terms of causality measures implies that the causality decomposition stays the same if we condition on other variables. The middle panel of Table 5 shows that conditioning on the NIKKEI index does not change the causality decomposition for the DAX and the Dow-Jones index. This results holds also for higher lag orders up to lag $p = 5$ as we can see from the columns of Table 5. The size of the instantaneous causality proportion varies with the lag length p but the highest linear dependence measure is obtained for lag order $p = 5$. This is also the criterion we suggest to choose the lag order in causality models.
Interestingly, we obtain similar causality decompositions if we use the extended causality measures of section 2.4 for blocks of time series. In panel 3 of Table 5 we have listed the decomposition for the NIKKEI and the Hang Seng Index on the one side and the DAX and the Dow-Jones index on the other side and we see that the decomposition of the causality measures vary in the same way as the univariate measure in panel 1. This gives rise to the interpretation that certain patterns of information flows in stock markets are similar and might depend on the geographical region and/or certain time periods.

6 Conclusions

In this paper we have proposed a new set of causality measures which are based on the feedback measures of Geweke (1984) for the class of Bayesian ARX-$ARCHX$-M and $VARX$-$VARCHX$-M models. Unidirectional causality measures are defined as log Bayes factors between time series models with and without the 'causing' variable. All causality measures are based on log marginal likelihoods and informative prior distributions. Prior distributions for the mean and variance equations are chosen similar to the tightness prior of Litterman (1986) and the exact marginal likelihood values are calculated in the process of MCMC sampling as it was suggested by Chib and Jeliazkov (1999). A simpler version of Bayesian causality measures can be obtained using the concept of a fractional prior distribution as it is implied by the fractional Bayes factor approach of O'Hagan (1995).
The computation of Bayesian causality measures is accompanied the advantage that the order of the model can be chosen by the same marginal likelihood criterion. In total 5 models have to be estimated for each model order and it is clear that the computation time will depend on fast and efficient MCMC sampling algorithms. Furthermore, for complex model classes, like the $VARX$-$VARCHX$ and the $VARX$-$ARCHXM$ class, factorizations of the overall causality measures are possible to explore the size of the contributions of the ARX part, the $ARCHX$ part and the $ARXM$ part. The examples show that the $ARCHX$ part of stock returns accounts for the largest fraction in the causality factorization analysis. This means that for causality measures the volatility equation of the 3-dimensional time series of stock returns is more important than the mean equation (in a VAR-$VARCH$-M model). This shows that this new class of Bayesian causality measures can quantify spillover effects in compact measures and on a high level of dynamic information condensation. We found that volatility components which originated from foreign returns have more weight than volatility components generated in the home market. The 'volatility-to-mean' feedbacks are modeled by the class of $ARCH-M$ models and in terms of likelihood contributions their influence is rather small. In further applications of this approach it is possible to explore how other extensions of $VARX$-$VARCHX$ models like power exponential or asymmetric $ARCH$ models can be incorporated into this new class of causality measures.

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F\p	1	2	3	4	5
$F(x \to y\|z)$	0.04401	0.06365	0.05713	0.07492	0.08288
%	0.00474	0.00687	0.00615	0.00807	0.00894
$F(y \to x\|z)$	0.66333	0.69173	0.72277	0.70878	0.75242
%	0.07145	0.07470	0.07780	0.07638	0.08121
$F(x \cdot y\|z)$	8.57632	8.50514	8.51051	8.49563	8.43017
%	0.92380	0.91843	0.91605	0.91554	0.90985
$F(x, y\|z)$	9.28366	9.26051	9.29041	9.27933	9.26547

Ta le 1.Conditional causality measures for the daily returns of the Dow Jones index (y), DAX index (x) and the Nikkei index (z) for the period June 21, 1996 to June 22, 1998

k	p	q	AR_y	$AR_y X_x$	$ARCH_y$	$ARCH_y X_x$
1	0	0	-232.1339	-232.1339	-114.5241	-114.7281
1	1	1	-232.1339	-217.5267	-114.5241	-113.6103
2	1	1	-232.5680	-219.6521	-114.5241	-113.2237
1	2	1	-232.1339	-217.5267	-113.4434	-112.1736
1	1	2	-232.1339	-217.5267	-114.5241	-113.5167
2	2	2	-232.5680	-219.6521	-113.4434	-112.8949
k	p	q	$AR_y ARCH_y$	$AR_y ARCH_y X_x$	$AR_y X_x ARCH_y$	$AR_y X_x ARCH_y X_x$
1	0	0	-111.5262	-109.6382	-111.9823	-112.4853
1	1	1	-110.0541	-109.3460	-110.3127	-106.4282
2	1	1	-109.0351	-107.9152	-104.7545	-101.6730
1	2	1	-106.0944	-106.0835	-107.3547	-106.6386
1	1	2	-110.0541	-106.1628	-110.1121	-103.5557
2	2	2	-104.0288	-103.2208	-103.3296	-102.1274
k	p	q	$AR_y ARCH_y M_y$	$AR_y X_x ARCH_y X_x M_y$	$aAR_y X_x ARCH_y a$	
1	0	0	-109.4534	-107.0929	-120.8106	
1	1	1	-106.0795	-102.7627	-113.5605	
2	1	1	-100.1090	-99.3378	-110.6477	
1	2	1	-105.6941	-101.4441	-114.8344	
1	1	2	-102.0795	-101.1029	-106.4286	
2	2	2	-100.4930	-99.6767	-102.8313	

Ta le 2. The log marginal likelihood for the family of ARX-$ARCHX$-M models with y=Dow Jones and x=DAX from June 21, 1996 to June 22, 1998

k	p	q	AR_x	AR_xX_y	$ARCH_x$	$ARCH_xX_y$
1	0	0	-233.5727	-233.5727	-115.2684	-115.6729
1	1	1	-233.5727	-219.2425	-115.2684	-114.8299
2	1	1	-234.7819	-221.6723	-115.2684	-114.2182
1	2	1	-233.5727	-219.2425	-114.7852	-113.5631
1	1	2	-233.5727	-219.2425	-115.2684	-114.6715
2	2	2	-234.7819	-221.6723	-114.7852	-113.2162
k	p	q	AR_xARCH_x	$AR_xARCH_xX_y$	$AR_xX_yARCH_x$	$AR_xX_yARCH_xX_y$
1	0	0	-112.6712	-110.3512	-113.7182	-112.4853
1	1	1	-111.1278	-110.1266	-111.6372	-108.0761
2	1	1	-110.5721	-108.3416	-105.7821	-103.3090
1	2	1	-107.7182	-107.5626	-108.6837	-106.3315
1	1	2	-111.6278	-107.4251	-111.1378	-102.4200
2	2	2	-105.6157	-104.5162	-104.3241	-98.5970
k	p	q	$AR_xARCH_xM_x$	$AR_xX_yARCH_xX_yM_x$	$aAR_xX_yARCH_xa$	
1	0	0	-110.8291	-108.7288	-121.8473	
1	1	1	-106.4444	-103.0355	-114.1019	
2	1	1	-100.7556	-100.2025	-110.6170	
1	2	1	-106.0485	-101.6797	-110.3606	
1	1	2	-103.8189	-101.3728	-108.6782	
2	2	2	-102.1104	-100.3985	-105.5924	

Ta le 3. The log marginal likelihood for the family of ARX-$ARCHX$-M models with y=Dow Jones and x=DAX from June 21, 1996 to June 22, 1998

k	p	q	$VAR_{xy}ARCH_{xy}X_{xy}$	$VAR_{xy}VARCH_{xy}M_{xy}$
1	0	0	-212.7389	-212.7389
0	1	0	-197.3291	-197.3291
1	1	1	-201.1179	-197.0942
2	1	1	-192.7238	-190.1090
1	2	1	-202.6823	-196.8166
2	0	0	-198.2364	-198.2364
1	1	2	-203.7189	-195.5267
2	2	2	-200.5062	-191.3603

Ta le 4. The log marginal likelihood of $VAR_{xy}(k)$

Causality					
p	1	2	3	4	5
$x \to y$	-0.17355	-0.40338	-0.54314	-0.67842	-0.84531
%	0.05112	0.14004	0.21310	0.22694	0.20932
$y \to x$	-0.65887	-0.91027	-1.25132	-1.36421	-1.49865
%	0.19409	0.31602	0.49095	0.45635	0.37111
$(x \cdot y)$	-2.56217	-0.55788	-0.75429	-0.94671	-1.69423
%	0.75478	0.19368	0.29594	0.31669	0.41955
x, y	-3.39456	-2.88035	-2.54875	-2.98934	-4.03819
Conditional Causality					
p	1	2	3	4	5
$x \to y$	-0.25432	-0.38542	-0.53675	-0.74319	-0.96523
%	0.06361	0.18167	0.17373	0.18848	0.19195
$y \to x$	-0.84754	-0.94653	-1.57621	-1.94531	-2.18656
%	0.21201	0.44616	0.51018	0.49336	0.43484
$(x \cdot y)$	-2.89567	-0.78953	-0.97654	-1.25446	-1.87655
%	0.72436	0.37216	0.31608	0.31815	0.37319
x, y	-3.99753	-2.12148	-3.08950	-3.94296	-5.02834
Extended Causality					
p	1	2	3	4	5
$X \to Y$	-0.45325	-0.12433	-0.48763	-0.57643	-0.78654
%	0.09163	0.04466	0.12313	0.12253	0.14001
$Y \to Y$	-0.95634	-1.67545	-1.78932	-2.23844	-2.59393
%	0.19334	0.60190	0.45182	0.47584	0.46175
$(X \cdot Y)$	-3.53662	-0.98382	-1.68328	-1.88928	-2.23712
%	0.71501	0.35343	0.42504	0.40161	0.39823
X, Y	-4.94621	-2.78360	-3.96023	-4.70415	-5.61759

Ta le 5 Causality analysis of Dow Jones and DAX from June 21, 1996 to June 22, 1998 $AR(p)$ model with informative prior (y=Dow Jones, x=DAX, z=Nikkei and [Y=(Dow Jones, DAX), X=(Nikkei, Hang Seng)] for equation (6), (7) and (8)

model	k=2, p=1, q=1, r=1			
	$x \to y$	$y \to x$	$x \cdot y$	(x, y)
$\tilde{R}(xARCHXM \to y)$	-133.2302	-134.5794	-9.4313	-277.2410
row %	0.4806	0.4854	0.0340	1.0000
column %	1.0000	1.0000	1.0000	1.0000
$\tilde{R}(x \to y)$	-12.9160	-13.1095	-243.0881	-269.1136
row %	0.0480	0.0487	0.9033	1.0000
			-1.3039	-269.1136
column %	0.0969	0.0974	0.1383	0.9707
$\tilde{R}(xARCH_{xy} \to y \vert x)$	-117.9790	-118.3634	-12.2582	-248.6006
row %	0.4746	0.4761	0.0493	1.0000
			-5.5125	-5.5125
column %	0.8855	0.8795	0.5845	0.0199
$\tilde{R}(xM_{xy} \to y \vert xyARCH_{xy})$	-2.3352	-3.1065	-9.4313	-14.8730
row %	0.1570	0.2089	0.6341	1.0000
			-2.6149	-2.6149
column %	0.0175	0.0231	0.2773	0.0094

Table 6. Triple xARCH-M factorization (AR start) with y=Dow Jones and x=DAX

model	k=2, p=1, q=1, r=1			
	$x \to y$	$y \to x$	$x \cdot y$	(x, y)
$\tilde{R}(xARCHXM \to y)$	-133.2302	-134.5794	-9.4313	-277.2410
row %	0.4806	0.4854	0.0340	1.0000
column %	1.0000	1.0000	1.0000	1.0000
$\tilde{R}(ARCH_{xy} \to y)$	-119.3444	-120.5637	-29.2055	-269.1136
row %	0.4435	0.4480	0.1085	1.0000
			-2.2113	-269.1136
column %	0.8958	0.8959	0.2345	0.9707
$\tilde{R}(x \to y \vert ARCH_{xy})$	-11.5506	-10.9092	-12.2582	-34.7180
row %	0.3327	0.3142	0.3531	1.0000
			-4.6052	-5.5125
column %	0.0867	0.0811	0.4883	0.0199
$\tilde{R}(xM_{xy} \to y \vert xyARCH_{xy})$	-2.3352	-3.1065	-9.4313	-14.8730
row %	0.1570	0.2089	0.6341	1.0000
			-2.6149	-2.6149
column %	0.0175	0.0231	0.2773	0.0094

Table 7. Triple xARCH-M factorization (ARCH start) with y=Dow Jones and x=DAX

	$x \to y$	$y \to x$	$x \cdot y$	(x, y)
AR	$\frac{AR_y}{AR_y X_x}$	$\frac{AR_x}{AR_x X_y}$	$\frac{AR_y X_x \cdot AR_x X_y}{VAR_{xy}}$	$\frac{AR_y \cdot AR_x}{VAR_{xy}}$
$ARCH$	$\frac{AR_y}{AR_y X_x ARCH_y}$	$\frac{AR_x}{AR_x X_y ARCH_x}$	$\frac{AR_y X_x ARCH_y \cdot AR_x X_y ARCH_x}{VAR_{xy} ARCH_{xy}}$	$\frac{AR_y \cdot AR_x}{VAR_{xy} ARCH_{xy}}$
$ARCH\text{-}M$	$\frac{AR_y}{AR_y X_x ARCH_y M_x}$	$\frac{AR_x}{AR_x X_y ARCH_x M_y}$	$\frac{AR_y X_x ARCH_y M_x \cdot AR_x X_y ARCH_x M_y}{VAR_{xy} ARCH_{xy}}$	$\frac{AR_y \cdot AR_x}{VAR_{xy} ARCH_{xy} M}$

Figure 1. Overview: Bayesian causality measures

	$x \to y$	$y \to x$	$x \cdot y$	(x, y)
$R(xARCHXM \to y)$ row-sum	$\frac{AR_y}{AA_{yx}M}$	$\frac{AR_x}{AA_{xy}M}$	$\frac{AA_{yx}M \cdot AA_{xy}M}{VV_{xy}M}$	$\frac{AR_y \cdot AR_x}{VV_{xy}M}$
$R(x \to y)$ row column	$\frac{AR_y}{AR_y X_x}$	$\frac{AR_x}{AR_x X_y}$	$\frac{AR_y X_x \cdot AR_x X_y}{VAR_{xy}}$ $\frac{AA_{yx}M \cdot AA_{xy}M}{VV_{xy}}$	$\frac{AR_y \cdot AR_x}{VAR_{xy}}$ $\frac{AR_y \cdot AR_x}{VAR_{xy}}$
$R(xARCH_y \to y \vert x)$ row column	$\frac{AR_y X_x}{AA_{yx}}$	$\frac{AR_x X_y}{AA_{xy}}$	$\frac{AA_{yx} \cdot AA_{xy}}{VV_{xy}}$ $\frac{VAR_{xy}}{VV_{xy}}$	$\frac{AR_y X_x \cdot AR_x X_y}{VV_{xy}}$ $\frac{VAR_{xy}}{VV_{xy}}$
$R(xM_{xy} \to y \vert xyARCH_{xy})$ row column	$\frac{AA_{yx}}{AA_{yx}M}$	$\frac{AA_{xy}}{AA_{xy}M}$	$\frac{AA_{yx}M \cdot AA_{xy}M}{VV_{xy}M}$ $\frac{VV_{xy}}{VV_{xy}M}$	$\frac{AA_{yx} \cdot AA_{yx}}{VV_{xy}M}$ $\frac{VV_{xy}}{VV_{xy}M}$

Figure 2. Two-way triple causality factorization

Univariate causality $x \to y$	$\frac{R(x\to y)=AR_y}{AA_{yx}M}$	$= \frac{AR_y}{AR_y X_x} \cdot \frac{AR_y X_x}{AA_{yx}} \cdot \frac{AA_{yx}}{AA_{yx}M}$
Univaraite causality $y \to x$	$R(y \to x) = \frac{AR_x}{AA_{xy}M}$	$= \frac{AR_x}{AR_x X_y} \cdot \frac{AR_x X_y}{AA_{xy}} \cdot \frac{AA_{xy}}{AA_{xy}M}$
Instandaneous causality $x \cdot y$ (inconsistent and consistent factors)	$R(x \cdot y) = \frac{AA_{yx}M \cdot AA_{xy}M}{VV_{xy}M}$	$\neq \frac{AR_y X_x \cdot AR_x X_y}{VAR_{xy}} \cdot \frac{AA_{yx} \cdot AA_{xx}}{VV_{xy}} \cdot \frac{AA_{xy}M \cdot AA_{xy}M}{VV_{xy}M}$ $= \frac{AA_{xy}M \cdot AA_{xy}M}{VAR_{xy}} \cdot \frac{VAR_{xy}}{VV_{xy}} \cdot \frac{VV_{xy}}{VV_{xy}M}$
Linear dependence (x, y)	$R(x, y) = \frac{AR_y \cdot AR_x}{VV_{xy}M}$	$= \frac{AR_y \cdot AR_x}{VAR_{xy}} \cdot \frac{VAR_{xy}}{VV_{xy}} \cdot \frac{AA_{yx}M \cdot VV_{xy}}{VV_{xy}M}$ $\neq \frac{AR_y \cdot AR_x}{VAR_{xy}} \cdot \frac{AR_y X_x \cdot AR_x X_y}{VV_{xy}} \cdot \frac{AA_{yx} \cdot AA_{xy}}{VV_{xy}M}$

Figure 3. Triple causality factorization