Solving Heterogeneous Agent Models: Introduction

Michael Reiter

Institute for Advanced Studies, Vienna

Heterogenous Agent Models in Macroeconomics
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1. Introduction

2. Krusell/Smith models

3. An Abstract Heterogeneous Agent Model

4. Some mathematical tools
   - Cross-sectional distributions
   - Fixed-point methods
   - Automatic Differentiation

5. The Krusell/Smith method

6. Approximation errors

7. Miscellaneous methods for Heterogeneous Agent Models
   - Explicit aggregation
   - Perturbation
**Outline**

1. **Introduction**
2. **Krusell/Smith models**
3. **An Abstract Heterogeneous Agent Model**
4. **Some mathematical tools**
   - Cross-sectional distributions
   - Fixed-point methods
   - Automatic Differentiation
5. **The Krusell/Smith method**
6. **Approximation errors**
7. **Miscellaneous methods for Heterogeneous Agent Models**
   - Explicit aggregation
   - Perturbation
Why study HA models?

- Analyzing policy design (unemployment insurance, etc.), typically in steady state.
- Explaining wealth distribution, for example through
  - uninsurable idiosyncratic income risk
  - entrepreneurship
  - return risk on assets, limited diversification of wealth
  - differences in discount factors
- Distributional consequences of fiscal and monetary policy,
- Welfare costs of business cycles
  Imperfect unemployment insurance.
- Aggregate elasticities from aggregating individual decisions.
  Example: aggregate labor supply (Chang and Kim 2006; Chang and Kim 2007)
- Asset trade and asset pricing
Main technical problem

High dimensionality of the state space

- Impossible to solve agents’ optimization problem precisely.
- Impossible to solve exactly the dynamics of high-dimensional economic system.
Exact aggregation

There are heterogenous agents models that allow exact aggregation:

- (Angeletos 2007): Aggregation over entrepreneurial households
  - no labor income risk
  - no exogenous borrowing constraint (only natural b.c.)
  - constant returns to scale to production of entrepreneurs

- Competitive search models, block recursivity (Menzio/Shi)
- Calvo pricing
- etc. ...
Approximate aggregation

means that aggregate law of motion can be well represented by a Markov process in a small number of observables.

Question: sources and limits of approximate aggregation?
Some representative models

- Models with a finite number of types of agents
  1. New Keynesian Models with Taylor price staggering
  2. OLG models
- The stochastic neoclassical growth model with heterogeneous households (Krusell and Smith 1998); and with portfolio choice (Krusell and Smith 1997)
- Models of aggregate labor supply (Chang and Kim 2006; Chang and Kim 2007)
- Models of lumpy investment (Thomas 2002)
- Models of state dependent pricing (Dotsey et al. 1999; Costain and Nakov 2011)
- Models of lumpy investment and state dependent pricing (Reiter, Sveen, and Weinke 2013)
- Distribution effects of monetary policy (Gornemann et al. 2014)
- Shocks to aggregate uncertainty (Bloom 2009)
- Dispersed information and stock prices (Mertens 2011)
Classes of Models: finite vs. infinite number of types

1. Finite (not too large) number of types
   - OLG models
   - Models with Taylor staggering (perhaps finite idiosyncratic productivity)

   Can be handled by perturbation or projection methods without resorting to state aggregation.

2. Infinite (or very large) number of types
   - Households with idiosyncratic income shocks
   - Firm models with several states (State-dependent pricing, lumpy investment, idiosyncratic productivity).

   This requires either
   - state aggregation, or
   - linearization in aggregate variables.
Classes of Models: convex vs. non-convex optimization problem of agents

1. Convex problem:
   - continuous choice
   - solution based on first order conditions (Euler equations)
   Can be solved with high precision by smooth approximation methods (projection methods).

2. Nonconvex: discrete choice, perhaps smoothing by stochastic adjustment costs (Dotsey, King, and Wolman 1999)
   May require discrete dynamic programming.
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Basic Assumptions

Firm side: neoclassical
- Many perfectly competitive firms
- Competitive factor markets
- Aggregate technology shock: AR(1) in TFP $z_t$

Households
- Continuum of ex-ante identical households
- Ex-post heterogeneity
  - Idiosyncratic productivity shocks
  - Differences in time discount factor, very persistent
- Incomplete markets:
  - No insurance for idiosyncratic shocks
  - In most model variants, imperfect insurance for aggregate shocks: only one asset (physical capital).
Assumption: Household optimize based on a very simple model of the aggregate economy

\[ K_t = \gamma_0 + \gamma_1 K_{t-1} + \gamma_2 z_t \]  

[With exogenous aggregate labor, \( K \) determines factor prices.] Coefficients \( \gamma \) are estimated by OLS from large sample

Celebrated finding: the simple model (1) has an \( R^2 \) of around 0.999999. Households do not make a significant mistake. Was confirmed in many variants of this model.

Question: is this a reasonable model of “bounded rationality”?
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Model setup

- Continuum (mass 1) of agents, ex ante identical. Index an agent by subscript $h$.
- Individual endogenous state: $x_{h,t} \in \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N\}$.
- Individual exogenous shock: $y_{h,t} = \bar{y}_j$ with probability $p_j$, $j = 1, \ldots, J$.
  $y_{h,t}$ is i.i.d. across agents and time.
- Agents differ ex post because of individual history $y_{h,t}$.
- Aggregate exogenous shock: $z_t$. Finite Markov chain with transition probability matrix $\Pi$.
- Decisions are taken after $z_t$, but before $y_{h,t}$ is realized.
Agents’ objective

\[
\max_{a_{h,t}} \mathbb{E} \sum_{t=0}^{\infty} \beta^t U(a_{h,t}, x_{h,t-1}, y_{h,t}, z_t, X_{t-1})
\]  

subject to \( a_{h,t} \in G(x_{h,t-1}) \) and

\[
\text{prob}(x_{h,t} = \bar{x}_j | x_{h,t-1} = \bar{x}_i) = P_{i,j}(a_{h,t}, x_{h,t-1}, y_{h,t}, z_t, X_{t-1})
\]

where

- \( X_t \equiv \sum_{i=1}^{N} \bar{x}_i \phi_{i,t} \)
- \( \phi_{i,t} \) is mass of agents at state \( \bar{x}_i \) at end of period \( t \).
- \( G(x_{h,t-1}) \) is the set of feasible actions \( a_{h,t} \) conditional on \( x_{h,t-1} \).
Structure of solution

- Aggregate State Space
  - $N$ aggregate state variables:
    1. $\phi_{i,t}$, $i = 1, \ldots, N$ with $N - 1$ degrees of freedom
    2. $z_t$

- Individual decision function

\[
a_{h,t} = A(x_{h,t-1}, \phi_{t-1}, z_t)
\] (4)
Aggregate Law of Motion

Given

- the individual decision function $\mathcal{A}(\cdot)$
- last period’s state $\vec{\phi}_{t-1}$
- this period’s realizations $(y_{h,t}, z_t)$ for all $h$

we get

$$\phi_{m,t} = \sum_{n=1}^{N} \sum_{j=1}^{J} \phi_{n,t-1} \cdot p_j \cdot P_{n,m}(\mathcal{A}(\bar{x}_n, \vec{\phi}_{t-1}, z_t), \bar{x}_n, \bar{y}_j, z_t, X_{t-1})$$ (5)

Denote ALM by $\vec{\Phi}(\vec{\phi}_{t-1}, z_t)$. 
Recursive equilibrium

- Agents’ optimization: Bellman equation

\[
V(\vec{x}_n, \vec{\phi}_{t-1}, z_t) = \max_a \left\{ \sum_j p_j \left[ U(a, \vec{y}_j, z_t, X_{t-1}) \right. \right. \\
+ \beta \sum_{m=1}^{N} P_{n,m}(a, \vec{x}_n, \vec{y}_j, z_t, X_{t-1}) E_t V\left(\vec{x}_m, \vec{\Phi}(\vec{\phi}_{t-1}, z_t), z_{t+1}\right) \left. \right\} \tag{6}
\]

which gives optimal decision \( A(\vec{x}_n, \vec{\phi}_{t-1}, z_t) \).

- \( \vec{\Phi}(\vec{\phi}_{t-1}, z_t) \), satisfies (5) given decision rule \( A(\vec{x}, \vec{\phi}, z) \).

- \( y_i \) and \( z \) follow exogenous processes

Notice: expectation \( E_t \) is over \( z_{t+1} \).
Assumptions for recursive equilibrium

- Full information: all agents observe cross-sectional distribution of $\chi$.
- Rational expectations (better: model-consistent expectations): agents use exact solution of model for forecasting future prices.
- Recursive structure: all decisions and equilibrium outcomes are time-invariant functions of state vector (in contrast to more general formulation: functions of history of shocks).
- Initial values: $z_0, \bar{\phi}_{-1}$. 
Approximate Recursive equilibrium

- Problem with recursive equilibrium: high-dimensional state space ($N$ aggregate states).
- Only aggregate variable affecting current return: $X_{t-1}$.
- Other aggregate variables only needed to predict future $X_t$. 
Assume simple process for $X_t$:

$$X_t - \mu = \rho(X_{t-1} - \mu) + \gamma z_t \tag{7}$$

Approximate consistency between PALM ("perceived aggregate law of motion"), Equ. (7), and individual behavior:

$\mu, \rho, \gamma$ in forecast function (PALM) identical to estimates obtained from long time series of model simulation.

Computation:

1. Choose parameters $\mu, \rho, \gamma$.
2. Solve household problem, given parameters
3. Simulate the economy (large sample of households)
4. Estimate parameters from simulated example
5. Iterate 1–3 until convergence.

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An Abstract Heterogeneous Agent Model


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Approximation II: Linearization around deterministic steady state (Reiter 2009b)

Consider big system of nonlinear equations:

- Markov process for $z_t$:
  \[ z_t = G(z_{t-1}, \varepsilon_t) \] (8)

- Distribution dynamics (5)

- FOC (assume it is sufficient)

\[
0 = \sum_j p_j \left[ U_a(a_n, \bar{y}_j, z_t, X_{t-1}) + \beta \sum_{m=1}^N \frac{\partial P_{n,m}(a_n, \bar{x}_n, \bar{y}_j, z_t, X_{t-1})}{\partial a_n} E_t V(\bar{x}_m, \phi_t, z_{t+1}) \right] \] (9)

This gives $2N + 1$ equations in $2N + 1$ variables: $a_n, n = 1, \ldots, N$; $\phi_n, n = 1, \ldots, N$; $z_t$. 
Linearization around deterministic steady state, ctd.

- Set all $z_t$ (but not the $y_{h,t}$!) to its unconditional expectation $z^*$.
- Assume there is a stationary distribution of $\vec{\phi}$ under optimal individual behavior when $z_t = z^*$, $\forall t$.
- Denote steady state values by $a_n^*$, $\phi_n^*$, $z^*$.
- Linearize (5), (9), (8) around steady state values, and solve by standard methods (possible up to around 10000 variables; for more variables: optimal state aggregation).
Approximation III: Perturbation around symmetric deterministic steady state (Mertens and Judd 2011)

- Set $z_t$ and the $y_{h,t}$ to their unconditional expectations $z^*$. and $y^*$.
- Compute higher-order perturbation around steady state $z^*$,
  
  $x_1^* = x_2^* = \ldots = x_N^*$, $a_1^* = a_2^* = \ldots = a_N^*$.
  
- Exploit symmetry of derivatives: $F_{i,j,k}$, $F_{j,i,k}$, $F_{i,j,k}$ etc.

The symmetry of derivatives makes it possible to compute higher-order perturbations for arbitrary $N$. 
Approximation IV: collocation method (Reiter 2009a)

Projection method: approximate decision rule \( A(x, \phi_1, \ldots, \phi_N, z) \) as linear combination of basis functions \( \phi_l \):

\[
A(x, \phi_1, \ldots, \phi_N, z) \approx \sum_{l=1}^{n_\gamma} \gamma_l \phi_l(x, \phi_1, \ldots, \phi_N, z)
\]  

(10)

- Using (5) and (8), plugging decision rule (10) into (9) gives residuals \( R(x, \phi_1, \ldots, \phi_N, z; \gamma) \).
- Collocation method requires to set \( R(\hat{x}^\alpha, \hat{\phi}_1^\alpha, \ldots, \hat{\phi}_N^\alpha, \hat{z}^\alpha; \gamma) = 0 \) at a predefined set of \( n_\gamma \) grid points:

\[
R(\hat{x}^\alpha, \hat{\phi}_1^\alpha, \ldots, \hat{\phi}_N^\alpha, \hat{z}^\alpha; \gamma) = 0, \quad \alpha = 1, \ldots, n_\gamma
\]  

(11)
Problem: because $N$ is large, $n_\gamma$ is huge except for linear approximation.

Possible solution: denote by $M_k(x)$ the $k$-th cross-sectional moment of $x$:

$$M_k(x) = \sum_{i=1}^{N} \phi_i \bar{x}_i^k$$ (12)

Approximate

$$A(x, \phi_1, \ldots, \phi_N) \approx \sum_{i_1, i_2, \ldots, i_K} \gamma_{i_1, i_2, \ldots, i_K} M_1^{i_1} M_2^{i_2} \cdots M_K^{i_K}$$ (13)
Collocation, choice of grid

- We have reduced parameters to a (relatively) small number.
- Now we need a small grid in a high-dimensional state space.
- Use simulation methods:
  1. Simulate model solution using a guess $\hat{\gamma}$.
  2. Learn from simulation about the part of the state space in which the economy spends most of the time.
  3. Choice some points from this part of the state space, and iterate. Example: Judd, Maliar, and Maliar (2012).
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Representing the cross-sectional distribution

1. Large sample of households
   - Advantage: simple, no systematic mistake
   - Disadvantage: slow, sampling error;

2. Parametric representation of distribution
   - Advantage: no sampling error, can be fast
   - Disadvantage: imposes functional form, which may bias results
   - Choices:
     - Finite number of point masses
     - Histogram (Reiter 2009c)
     - Linear combination of smooth distributions
     - Hybrid methods
Assume a finite grid \( \mathcal{X} = \{ \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N \} \).

The distribution at time \( t \) is represented by \( \phi_{i,t} \), the mass of agents at state \( \bar{x}_i \) in period \( t \).
Collect the probabilities in the vector \( \Phi \).

Denote by \( P_i(x) \) the probability of going from state \( \bar{x}_i \) to state \( x \):

\[
P_i(x) = \text{prob}(x_{t+1} = x | x_t = \bar{x}_i)
\]  

(14)

Here, \( x \) need not be on the grid \( \mathcal{X} \).
Transition functions on point masses

- If \( x \notin \mathcal{X} \), approximate this transition by two possible transitions:

\[
prob(x_{t+1} = \bar{x}_j | x_t = \bar{x}_i) = \begin{cases} 
  P_i(x) \frac{\bar{x}_{k+1} - x}{\bar{x}_{k+1} - \bar{x}_k} & \text{if } j = k \\
  P_i(x) \frac{x - \bar{x}_k}{\bar{x}_{k+1} - \bar{x}_k} & \text{if } j = k + 1 
\end{cases}
\]  

(15)

where \( \bar{x}_k \leq x < \bar{x}_{k+1} \).

- If the system can reach several points \( x \) starting from \( \bar{x}_i \), approximate this by a combination of transitions of type (15).

- Transitions of type (15) can be represented by a transition matrix \( \Pi \), such that

\[
\Phi_{t+1} = \Pi \Phi_t 
\]  

(16)

The invariant distribution is given by the eigenvector

\[
\Phi^* = \Pi \Phi^* 
\]  

(17)
Transition functions on point masses

- Advantage of this representation:
  - Forward iteration is a linear operation
  - Invariant distribution is a standard problem of linear algebra: eigenvector of largest eigenvalue

- Disadvantage: artificial randomness (which grid point to reach) adds extra variance.
  Possible solution: more general transition matrix. Use transition to several grid points s.t.
  1. $m$ moments of next period’s conditional distribution are matched exactly
  2. other moments are fitted as well as possible
More general transition scheme

\[
\begin{align*}
\min_{p_j, \ j \in J} \ & \sum_{k=m+1}^{K} \left[ \left( \sum_{j \in J} p_j \bar{x}_j^k - \sum_x P_i(x)x^k \right) \right] \\
\text{s.t.} \ & \sum_{j \in J} p_j \bar{x}_j = \sum_x P_i(x)x \\
& \ldots = \ldots \\
& \sum_{j \in J} p_j \bar{x}_j^m = \sum_x P_i(x)x^m
\end{align*}
\] (18) (19)

- Set \( p_j = 0 \) for \( j \notin J \)
- choose \( J \) not too big, to have sparse transition matrix.
Modeling smooth distributions

We approximate a smooth distribution by

\[ f_t(x) = \sum_k \alpha_k(t) \phi_k(x), \quad \sum_k \alpha_k(t) = 1 \]  

(20)

where each \( \phi_k \)
- has bounded (ideally, small) support
- is a density function (non-negative, \( \int \phi_k(x) \, dx = 1 \))

Take as given a transition function \( \Pi(x, y) \).
We want to model the transition from initial density \( f_t(x) \) to the new density \( f_{t+1}(x) \).
Moments of smooth distributions

We will try to match $J$ moments, $j = 1, \ldots, J$, of the new density:

$$m_{j,t+1} = \int_{x'} H_j(x') f_{t+1}(x') dx'$$  \hspace{1cm} (21)

Define

$$h_{j,k} \approx \int H_j(x) \phi_k(x) dx$$  \hspace{1cm} (22)

$$h'_{j,k} \approx \int_{x'} H_j(x') \left( \int \Pi(x, x') \phi_k(x) dx \right) dx'$$  \hspace{1cm} (23)

Then

$$m_{j,t+1} = \int_{x'} H_j(x') \left( \int \Pi(x, x') f_t(x) dx \right) dx' = \sum_k \alpha_k(t) h'_{j,k}$$  \hspace{1cm} (24)
Transition of smooth distributions

Given \( f_t \) as time \( t \) distribution and with transition function \( \Pi(x, x') \), choose distribution at \( t + 1 \) as in (20) with \( \alpha_k \) to solve

\[
\min_{\alpha_1, \ldots, \alpha_K} \sum_{j=J_0+1}^J \omega_j \left( \sum_k \alpha_k h_{j,k} - m_{j,t+1} \right)^2
\]

subject to

\[
\sum_k \alpha_k = 1 \quad \text{(25b)}
\]

\[
\alpha_k \geq 0, \quad \forall k \quad \text{(25c)}
\]

\[
\sum_k \alpha_k h_{j,k} = m_{j,t+1}, \quad j = 1, \ldots, J_0 \quad \text{(25d)}
\]

\( J_0 \) moments are matched exactly. The \( \omega_j \) are weights.
Invariant smooth distributions

Find $\alpha_k$ such that

$$(\alpha_1, \ldots, \alpha_K) = \arg\min_{\alpha_1, \ldots, \alpha_K} \sum_{j=J_0+1}^{J} \omega_j \left( \sum_k \alpha_k h_{j,k} - \sum_k \alpha_k h'_{j,k} \right)^2$$  (26a)

subject to

$$\sum_k \alpha_k = 1$$  (26b)
$$\alpha_k \geq 0, \quad \forall k$$  (26c)
$$\sum_k \alpha_k h_{j,k} = \sum_k \alpha_k h'_{j,k} \quad j = 1, \ldots, J_0$$  (26d)
Distributions, evaluation

- Distribution as large sample: simple, unbiased, slow.
- Discrete approximation: potential bias; all operations based on linear algebra.
- Smooth approximation: good to cover smooth distributions over wide support. Needs nonlinear operations (quadratic programming problem, nonlinear root finding).
Global solution: find the fixed point

- Quasi-Newton methods: solve one big system of nonlinear equations (all residuals at all grid points)
- Time iteration (as in dynamic programming):
  1. Use parameters $\gamma_{t+1}$ to approximate variables in period $t + 1$.
  2. Separately for each grid point: solve equation system for time-$t$ variables.
  3. Use time-$t$ variables to update approximation parameters $\gamma_t$.
  4. Iterate until convergence.
- Fixed point iteration (Judd, Maliar, Maliar, and Valero 2013).
  1. Guess parameter vector.
  2. Simulate model, using this guess
  3. Update parameter vector from simulation.
Newton vs. Time-Iteration

- **Quasi-Newton:**
  - quadratic convergence
  - requires solution of system of $n$ linear equations ($n$ is dimension of parameter vector $\gamma$).
    - computation is of order $n^3$ (solving for Newton step)
    - memory is of order $n^2$

- **Time iteration**
  - local convergence
  - computation is of order $n^2$
  - memory is of order $n$.

For $n$ very large, quasi-Newton may not be feasible.
Convergence of fixed point iteration

Assume the dampened fixed point iteration

\[ p_{k+1} = p_k + \mu(\hat{p}_k - p_k) \]  \hspace{1cm} (27)

Then

\[ p_{k+1} - p^* = (p_k - p^*) + \mu(\hat{p}_k - p^* - (p_k - p^*)) \]  \hspace{1cm} (28)

For \( p_{k+1} - p^* \) in a neighborhood of \( p^* \), \( ||p_{k+1} - p^*|| < ||p_k - p^*|| \) if \( \rho(I + \mu(J - I)) < 1 \), where \( J \) is the Jacobian of the mapping \( p \rightarrow \hat{p} \) at \( p^* \). This is the case for sufficiently small \( \mu \) iff all the eigenvalues of \( (J - I) \) have negative real part.
Automatic Differentiation

- Numerical computation of derivatives of a function \( f(x) \) at a specific point \( x_0 \), using the exact rules of differentiation.
- Example:

\[
f(x) = \log(x \cdot \sin(x^3));
\]

\[
a = x^3; \quad b = \sin(a); \quad c = x \cdot b; \quad f(x) = \log(c)
\]

Then, at \( x = x_0 \),

\[
da/dx = 3 \cdot x_0^2 \quad \text{(30a)}
\]

\[
db/dx = \cos(a) \cdot da/dx \quad \text{(30b)}
\]

\[
dc/dx = b + x_0 \cdot db/dx \quad \text{(30c)}
\]

\[
df/dx = (1/c) \cdot dc/dx \quad \text{(30d)}
\]

These computations can be automatized by the computer in object-oriented programming through operator overloading.
Efficiency of Automatic Differentiation

1. Theoretical result: computing the complete gradient of a function $f : \mathbb{R} \rightarrow \mathbb{R}^n$ takes not more than 5 times the operation that it takes to compute $f$. But this is difficult to implement (reverse mode).

2. The chain of calculations in (30) is easy to implement, but often rather inefficient.

3. In our application, even forward mode rather efficient, because
   - use of implicit function theorem in two-step approximation
   - function evaluation:
     $$\frac{\partial A(x)}{\partial \gamma} = \frac{\partial A(x)}{\partial x} \frac{\partial x}{\partial \gamma}$$

Since $x$ has much fewer elements than $\gamma$, (31) is much faster than computing $\frac{\partial A(x)}{\partial \gamma}$ by forward differentiation!
Outline

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3. An Abstract Heterogeneous Agent Model
4. Some mathematical tools
   - Cross-sectional distributions
   - Fixed-point methods
   - Automatic Differentiation
5. The Krusell/Smith method
6. Approximation errors
7. Miscellaneous methods for Heterogeneous Agent Models
   - Explicit aggregation
   - Perturbation
Krusell/Smith method: Overview

1. Postulate an aggregate law of motion “ALM”:
   \[ X_t = \mathcal{A}(X_{t-1}, Z_{t-1}, Z_t; \theta) \]  
   (32)
   for aggregate states (\( X \): endogenous, \( Z \): exogenous), which is parameterized by \( \theta \).

2. Fix some initial \( \theta_0 \).

3. Given (32) and \( \theta_k \), solve for individual decision function:
   \[ d_i(X_t, Z_t, x_{t,i}, z_{t,i}; \theta) \]  
   (33)
   (\( x \) and \( z \): individual states).
   Using (33), simulate the aggregate economy given realizations for exogenous \( Z_t \) and \( z_{t,i} \).

4. Estimate \( \theta \) in (32) from the simulation. Denote the estimate by \( \hat{\theta}_{k+1} \). Update the parameters by
   \[ \theta_{k+1} = \mu \hat{\theta}_{k+1} + (1 - \mu) \theta_k \]  
   (34)
   with \( 0 < \mu < 1 \).
The vector of aggregate states \((X: \text{endogenous}, Z: \text{exogenous})\) must include all information relevant for economic agents (prices, aggregate demand etc.)

Realizations for the exogenous processes \(Z_t\) and \(z_{t,i}\) must be kept fixed across iterations, to allow convergence.

No theory on how to choose \(\mu\). Lower \(\mu\) makes convergence more likely, but slows down the algorithm. Convergence not guaranteed.

The aggregate vector \(X\) may contain equilibrium objects such as prices. When simulating the economy, we might want to solve for equilibrium in each period. Probably necessary if quantities depend on prices in a sensitive way (for example, asset choice).
Krusell/Smith: Details Matter

- Quasi-Newton method (to find parameters of ALM) has better convergence properties than FPI.
- Aggregate simulation: pseudo random numbers vs. low-discrepancy points (Sobol etc.); good results with latter in finance (Paskov 1994); tends to reduce sampling error.
- Representation of cross-sectional distribution: large sample of individuals contains sampling error. Consider using point masses, and forward cross-sectional distribution.
- Solving household problem: backward iteration, use Carroll’s endogenous fixed point method (Carroll 2006).
Function approximation

Function approximation:

- Smooth approximation in aggregate variables:

\[
    w = \mathcal{I}_A (\Phi; S) \quad (35)
\]

Multivariate polynomial approximation can be used.

- Two-step approximation for household variables:
  1. Get household parameters by interpolation in aggregate variables:

\[
    \phi = \mathcal{I}_A (\Phi; S) \quad (36)
\]

  2. Then interpolate household variables

\[
    c (S, x, z; \Phi) = \mathcal{I}_I (\phi; s) \quad (37)
\]

Spline approximation (special handling of kinks) is better than polynomials.
Krusell/Smith: Conceptual issues

- Functional form of ALM not consistent with individual decisions
- IRs depend on shock variances in two ways:
  - Uncertainty affects behavior (feature of the model)
  - Signal extraction problem (feature of the solution concept)
- Not structural: forecast functions based on history, not immune to Lucas critique.
- In NK models: forecast functions may rule out indeterminacy.
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Types of approximation error

There are (at least) three types of approximation error:

1. Error from discretization, coming from
   - finite approximation of cross-sectional distribution
   - finite approximation of decision functions
   These errors are already present in steady state; they can be estimated by increasing the size of the discrete approximation.

2. Error from approximation in aggregate variables (example: linearization);
   can be estimated by computing higher order approximations.

3. Error from aggregation;
   can be avoided in linearized model by state reduction.
Approximation errors

### Accuracy measure I: unconditional RMSE

\[ z_t = A^* z_{t-1} + B^* \epsilon_t \]  
\[ z \equiv \begin{bmatrix} x_t \\ \hat{x}_t \end{bmatrix}, \quad A^* \equiv \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^* \equiv \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \]  
\[ \Sigma_z = A^* \Sigma_z A^{*T} + B^* \Sigma_\epsilon B^{*T} \]

RMSE of prediction of \( y_i \):

\[
\sqrt{\left[ \begin{array}{c} C_i - \hat{C}_i \\ \end{array} \right] \Sigma_z \left[ \begin{array}{c} C'_i \\ \hat{C}'_i \end{array} \right]} \]

\[
\sqrt{\frac{C_i \Sigma_x C'_i}{C_i \Sigma_x C'_i}}
\]
Approximation errors

Accuracy measure II: difference impulse response

\[
\frac{\max_t |y_{i,t;j} - \hat{y}_{i,t;j}|}{\max_t |y_{i,t;j}|}
\]

- \(y_{i,t;j}\): impulse response at \(t\) of variable \(i\) to shock \(j\), exact model
- \(\hat{y}_{i,t;j}\): same for reduced model
Accuracy measure III: maximal error in transition path

\[
\max_{t=1}^{T} \| H A^t - \hat{A}^t H \| \quad (43)
\]
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Explicit aggregation I

Individual decision:

\[ k'(k, s) = \sum_{i=0}^{l} \psi_i(s) k^i \]  \hspace{1cm} (44)

Aggregation, first moment:

\[ M'_1 \equiv K'(s) \equiv \int k'(k, s) \, dF(k) \]  \hspace{1cm} (45)

\[ = \sum_{i=0}^{l} \psi_i(s) \int k^i \, dF(k) \]  \hspace{1cm} (46)

\[ = \sum_{i=0}^{l} \psi_i(s) M_i \]  \hspace{1cm} (47)

\( M_i, i = 1, \ldots, l \) are state variables and need to be predicted.
Explicit aggregation II

Aggregation, second moment:

\[ M'_2 \equiv \int [k'(k, s)]^2 dF(k) \]  
(48)

\[ = \int \left( \sum_{i=0}^{l} \psi_i(s)k^i \right)^2 dF(k) \]  
(49)

\[ = \sum_{i=0}^{2l} \hat{\psi}_i(s)M_i \]  
(50)

Moments of order up to \( 2l \) are needed.
Infinite regress.
Explicit aggregation: avoiding infinite regress

Approximate

\[ [k'(k, s)]^j \approx \sum_{i=0}^{I} \psi_{j,i}(s)k^i \] (51)

Then

\[ M'_j \equiv \int [k'(k, s)]^j \, dF(k) = \sum_{i=0}^{I} \psi_{j,i}(s) \int k^i \, dF(k) \] (52)

\[ = \sum_{i=0}^{I} \psi_{j,i}(s)M_i \] (53)

Get the \( \psi_{j,i}(s) \) from the \( \psi_{i}(s) \) by projection methods.
Explicit aggregation: computing the $\Psi_{j,i}$

Aim: determine $\psi_{j,i}(s)$ such that

$$\int [k'(k, s)]^j dF(k) = \int \left( \sum_{i=0}^{l} \psi_i(s) k^i \right)^j dF(k)$$

$$\approx \int \sum_{i=0}^{l} \psi_{j,i}(s) k^i dF(k)$$

without knowing $F(k)$.

Is this possible? A $l$-th degree polynomial can be a good approximation to the $(j \cdot l)$-th degree polynomial over a certain range.
Explicit aggregation: projection method

Projection (weighted residual) method: use weighting functions $\omega_j(k)$ such that

$$\int \left[ \left( \sum_{i=0}^{l} \psi_i(s)k^i \right)^j - \sum_{i=0}^{l} \psi_{j,i}(s)k^i \right] \omega_l \, dk = 0$$
Explicit aggregation: choosing residual functions

How to choose the $\omega_l$?

- Den Haan and Rendahl (2010):
  - In general, not very explicit about this.
  - Application: extremely simple, no exact aggregation even for first moment.

- Why not use *some* information about cross-sectional distribution, for example steady state $F(k)$?
Explicit aggregation: general basis functions

Individual decision:

\[
k'(k, s) = \sum_{i=0}^{l} \psi_i(s) B_i(k) \quad (54)
\]

Aggregation, first moment:

\[
M'_1 \equiv K'(s) \equiv \int k'(k, s) \, dF(k) \quad (55)
\]

\[
= \sum_{i=0}^{l} \psi_i(s) \int B_i(k) \, dF(k) \quad (56)
\]

\[
= \sum_{i=0}^{l} \psi_i(s) M_i^B \quad (57)
\]

It is convenient to use \( B_1(k) = k \), so that \( M_1 = M_1^B \). (Aggregate capital is needed for prices.)

\( M_i, i = 1, \ldots, l \) are state variables and need to be predicted.
Explicit aggregation: avoid inf. regress, general case

Approximate

\[ B_j(k'(k, s)) \approx \sum_{i=0}^{l} \psi_{j,i}(s)B_i(k) \]  \hfill (58)

Then

\[ M_j^{B'} \equiv \int B_j(k'(k, s)) \, dF(k) = \sum_{i=0}^{l} \psi_{j,i}(s) \int B_i(k) \, dF(k) \]  \hfill (59)

\[ = \sum_{i=0}^{l} \psi_{j,i}(s)M_i^{B} \]  \hfill (60)

Again, get the \( \psi_{j,i}(s) \) from the \( \psi_i(s) \) by projection methods.
Explicit aggregation: reducing number of states

Above approach uses as many aggregate states as individual basis functions.
Could be simplified, for example:

$$M'_1 \approx k'(M_1, s) = \sum_{i=0}^{l} \psi_i(s)B_i(M_1)$$  \hspace{1cm} (61)

Only one state ($M_1$) needed; like approximate aggregation.
Explicit aggregation: summary

Could work well for models with smooth decisions.
Needs more experimentation.
Perturbation Solution, general

The solution \( y(x) \) depends parametrically on the standard deviation of shocks, \( \sigma \).
Write this as \( y(x; \sigma) \).
The perturbation approach approximates this as (scalar case)

\[
y(x; \sigma) \approx y^* + y_x(x^*; 0)(x - x^*) + y_\sigma(x^*; 0)\sigma \\
+ \frac{1}{2}y_{xx}(x^*; 0)(x - x^*)^2 + \frac{1}{2}y_{\sigma\sigma}(x^*; 0)\sigma^2 \\
+ y_{x\sigma}(x^*; 0)(x - x^*)\sigma \\
+ \ldots
\]
Advantages of Local (Perturbation) Methods

1. Can be obtained “mechanically” (just differentiate often enough at the deterministic steady state).
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3. Are difficult to implement, but
4. a toolkit is available: Dynare.
Problems with Perturbation Methods

Even if solution is analytic,

1. high-order approximation may be necessary to get sufficient accuracy;
Problems with Perturbation Methods

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Perturbation methods, simple model

(Mertens and Judd 2011)
There are $I$ individuals (many), ex ante identical.
Variables (in total: $3I + 1$):

- Individual states: $k_i$ (assets), $a_i$ (labor productivity, exogenous, AR 1), $i = 1, \ldots, I$.
- Aggregate states: $z$ (TFP, exogenous, AR 1).
- Decision variables: $c_i$

Equations (in total: $3I + 1$):

- Asset accumulation, for $i = 1, \ldots, I$.
- AR1 labor productivity, $i = 1, \ldots, I$.
- Euler equation, $i = 1, \ldots, I$.
- AR1 for TFP
Perturbation methods, compact notation

\[ E F(K, A, z, C(K, A, z), K', A', z', C(K', A', z')) = 0 \quad (62) \]

where capital letters mean vectors: \( K = (k_1, k_2, \ldots, k_I) \) etc.

**Algorithm**

1. Compute symmetric deterministic steady state (all shocks equal zero).
2. Differentiate equation system several times w.r.t. elements of \( K, A, z \) etc. Gives a huge equation system. Gives implicitly the derivatives of \( C \) w.r.t. state variables.
3. Exploit symmetry:

\[ \frac{\partial c_i}{\partial k_j} = \frac{\partial c_i}{\partial k_l}, \forall j \neq i, l \neq i \quad (63) \]

Similar for higher derivatives.
Result: $C$ is written not as a function of some “moments”, but as a function of all individual states. Manageable because of symmetry.

- Same idea could be used by projection methods.
- Results valid for small shocks (small aggregate and small idiosyncratic shocks).
- Cannot really handle borrowing constraints.
\[ c_i = c^* + \frac{\partial c_i}{\partial k_i}(k_i - k^*) + \sum_{j \neq i} \frac{\partial c_i}{\partial k_j}(k_j - k^*) + \frac{1}{2} \frac{\partial^2 c_i}{\partial k_i^2}(k_i - k^*)^2 \]

\[ + \frac{1}{2} \sum_{j \neq i} \frac{\partial^2 c_i}{\partial k_j^2}(k_j - k^*)^2 + \frac{1}{2} \sum_{j \neq i, l \neq i} \frac{\partial^2 c_i}{\partial k_j \partial k_l}(k_j - k^*)(k_l - k^*) + \ldots \]

\[ = c^* + \frac{\partial c_1}{\partial k_2} \sum_j (k_j - k^*) + \left( \frac{\partial c_1}{\partial k_1} - \frac{\partial c_1}{\partial k_2} \right)(k_i - k^*) \]

\[ + \frac{1}{2} \frac{\partial^2 c_1}{\partial k_2^2} \sum_{j, l} (k_j - k^*)(k_l - k^*) + \frac{1}{2} \left( \frac{\partial^2 c_1}{\partial k_1^2} - \frac{\partial^2 c_1}{\partial k_2^2} \right)(k_i - k^*)^2 \]

\[ + \frac{1}{2} \frac{\partial^2 c_1}{\partial k_2 \partial k_3} \sum_{j \neq i, l \neq i} (k_j - k^*)(k_l - k^*) + \ldots \]

\[ = c^* + \frac{\partial c_1}{\partial k_2} M_1(k) + \left( \frac{\partial c_1}{\partial k_1} - \frac{\partial c_1}{\partial k_2} \right)(k_i - k^*) \]

\[ + \frac{1}{2} \frac{\partial^2 c_1}{\partial k_2^2} M_2(k) + \ldots \]
The pruned state-space system for non-linear dsge models: Theory and empirical applications.

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The impact of uncertainty shocks.  

The method of endogenous gridpoints for solving dynamic stochastic optimization problems.  


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Manuscript.


